

SYMMETRY IN THE PAINLEVÉ SYSTEMS AND THEIR EXTENSIONS TO FOUR-DIMENSIONAL SYSTEMS

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ABSTRACT. We give a new approach to the symmetries of the Painlevé equations P_V, P_{IV}, P_{III} and P_{II} , respectively. Moreover, we make natural extensions to fourth-order analogues for each of the Painlevé equations P_V and P_{III} , respectively, which are natural in the sense that they preserve the symmetries.

0. INTRODUCTION

This is the third paper in a series of four papers (see [13, 14]), aimed at giving a complete study of the following problem:

Problem 0.1. *For each affine root system A with affine Weyl group $W(A)$, find a system of differential equations for which $W(A)$ acts as its Bäcklund transformations.*

At first, let us summarize the results obtained up to now in the following list.

| Type | System | Dimension | References |
|-------------|-------------------------------|-----------|------------|
| $A_4^{(1)}$ | 2-coupled Painlevé IV system | 4 | [4, 5, 13] |
| $A_5^{(1)}$ | 2-coupled Painlevé V system | 4 | [4, 5, 13] |
| $D_5^{(1)}$ | 2-coupled Painlevé V system | 4 | [14] |
| $D_6^{(1)}$ | 2-coupled Painlevé VI system | 4 | [13] |
| $B_4^{(1)}$ | 2-coupled Painlevé III system | 4 | [14] |

Each of them is a family of coupled Painlevé systems with the following form for its Hamiltonian:

$$\begin{aligned}
 (1) \quad & \frac{dx}{dt} = \frac{\partial H}{\partial y}, \quad \frac{dy}{dt} = -\frac{\partial H}{\partial x}, \quad \frac{dz}{dt} = \frac{\partial H}{\partial w}, \quad \frac{dw}{dt} = -\frac{\partial H}{\partial z}, \\
 & H(x, y, z, w, t; \alpha_0, \alpha_1, \dots, \beta_0, \beta_1, \dots) \\
 & = H_*(x, y, t; \alpha_0, \alpha_1, \dots) + H_*(z, w, t; \beta_0, \beta_1, \dots) + R \\
 & (* = VI, V, IV, III).
 \end{aligned}$$

Here the symbol R denotes the interaction term for each system.

Our idea is to find a system in the following way:

- (1) We make a set of invariant divisors given by connecting two copies of them given in the case of the Painlevé systems by adding the term with invariant divisor $x - z$.
- (2) We make the symmetry associated with a set of invariant divisors given by 1.
- (3) We make the holomorphy conditions r_i associated with the symmetry in 2.

- (4) We look for a polynomial Hamiltonian system with the holomorphy conditions given by 3.

The crucial idea of this work is to use the holomorphy characterization of each system, which can be considered as a generalization of Takano's theory [3, 16].

In the next stage, following the above results, we try to seek a system with $W(B_3^{(1)})$ -symmetry. At first, one might try to seek a system in dimension four with its symmetry. However, such a system can not be obtained. In this paper, we will change our viewpoint, by seeking a system not in dimension four but in dimension two. In dimension two, it is well-known that the Painlevé systems P_J , ($J = VI, V, IV, III, II, I$) have the affine Weyl group symmetries explicitly given in the following table.

| P_J | P_I | P_{II} | P_{III} | P_{IV} | P_V | P_{VI} |
|----------|-------|----------------|----------------|----------------|----------------|----------------|
| Symmetry | none | $W(A_1^{(1)})$ | $W(C_2^{(1)})$ | $W(A_2^{(1)})$ | $W(A_3^{(1)})$ | $W(D_4^{(1)})$ |

This paper is the stage in this project where we find a new viewpoint for the symmetries of the Painlevé equations P_V, P_{IV}, P_{III} and P_{II} , that is, we will show that each of the Painlevé equations P_V, P_{IV}, P_{III} and P_{II} has hidden affine Weyl group symmetry of types $B_3^{(1)}, G_2^{(1)}, D_3^{(2)}$ and $A_2^{(2)}$, respectively. We seek these symmetries for a Hamiltonian system in charts other than the original chart in each space of initial conditions constructed by K. Okamoto. In other charts, we can find hidden symmetries different from the ones in the original charts. Furthermore, in the case

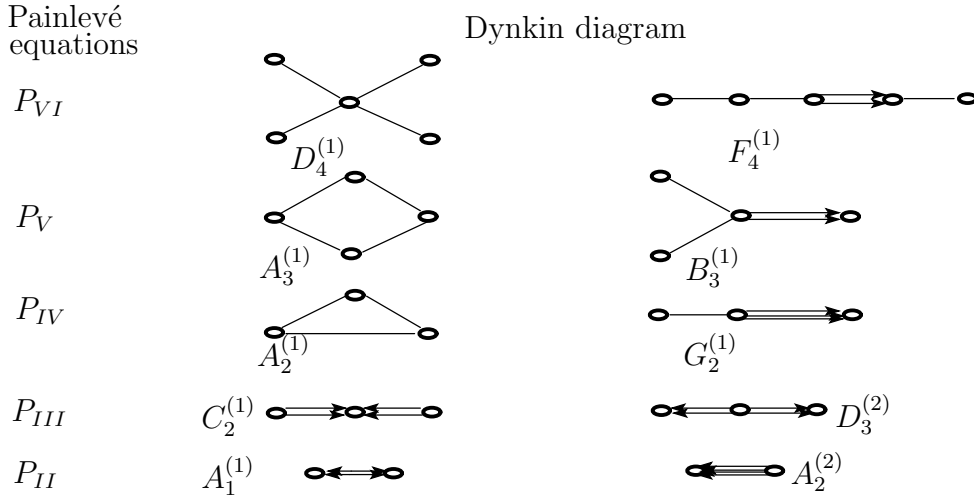


FIGURE 1. Symmetries of the Painlevé equations

of dimension four we make natural extensions for each of the Painlevé equations P_V and P_{III} , natural in the sense that they preserve the symmetries.

This paper is organized as follows. In Sections 1 through 4, we present two-dimensional polynomial Hamiltonian systems with $W(B_3^{(1)})$, $W(G_2^{(1)})$, $W(D_3^{(2)})$ and $W(A_2^{(2)})$ -symmetry, respectively. We will show that each system coincides with the Painlevé V (resp. IV, III, II) system. We also give an explicit confluence process from the system of type $D_4^{(1)}$ (resp. $B_3^{(1)}, B_3^{(1)}, G_2^{(1)}$) to the system of type $B_3^{(1)}$ (resp. $G_2^{(1)}, D_3^{(2)}, A_2^{(2)}$). In Sections 5 and 6, we present a family of coupled Painlevé V (resp. III) systems in dimension four with $W(B_5^{(1)})$ (resp. $W(D_5^{(2)})$)-symmetry. We also show that this system coincides with a family of coupled Painlevé V (resp. III)

systems in dimension four with $W(D_5^{(1)})$ (resp. $W(B_4^{(1)})$)-symmetry (see [14]). In the final section, we propose further problems on Problem 0.1.

1. THE SYSTEM OF $B_3^{(1)}$

In this section, we present a 3-parameter family of two-dimensional polynomial Hamiltonian systems given by

$$(2) \quad \begin{cases} \frac{dx}{dt} = \frac{-2x^3y + 2x^2y - (\alpha_1 + 2\alpha_2)x^2 - (t - 1 + 2\alpha_3)x + t}{t}, \\ \frac{dy}{dt} = \frac{3x^2y^2 - 2xy^2 + 2(\alpha_1 + 2\alpha_2)xy + (t - 1 + 2\alpha_3)y + (\alpha_1 + \alpha_2)\alpha_2}{t} \end{cases}$$

with the polynomial Hamiltonian (cf. [2, 6])

$$(3) \quad \begin{aligned} H_{B_3^{(1)}}(x, y, t; \alpha_1, \alpha_2, \alpha_3) \\ = -\frac{-ty + x^3y^2 - x^2y^2 + (\alpha_1 + 2\alpha_2)x^2y + (t - 1 + 2\alpha_3)xy + (\alpha_1 + \alpha_2)\alpha_2x}{t}. \end{aligned}$$

Here x and y denote unknown complex variables and $\alpha_0, \alpha_1, \alpha_2$ and α_3 are complex parameters satisfying the relation:

$$(4) \quad \alpha_0 + \alpha_1 + 2\alpha_2 + 2\alpha_3 = 1.$$

We note that the Hamiltonian system (2) is a polynomial in the canonical variables x, y . In this sense, we call the system (2) as a polynomial Hamiltonian system.

Theorem 1.1. *The system (2) admits extended affine Weyl group symmetry of type $B_3^{(1)}$ as the group of its Bäcklund transformations (cf. [6]), whose generators are explicitly given as follows:*

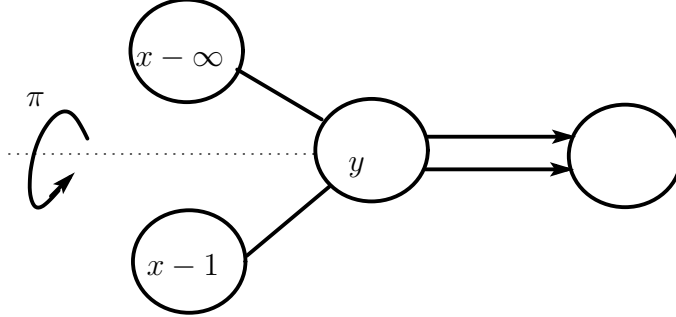


FIGURE 2. Dynkin diagram of type $B_3^{(1)}$

$$(5) \quad \begin{aligned} s_0 : (x, y, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3) &\rightarrow (x, y - \frac{\alpha_0}{x-1}, t; -\alpha_0, \alpha_1, \alpha_2 + \alpha_0, \alpha_3), \\ s_1 : (x, y, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3) &\rightarrow (x, y, t; \alpha_0, -\alpha_1, \alpha_2 + \alpha_1, \alpha_3), \\ s_2 : (x, y, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3) &\rightarrow (x + \frac{\alpha_2}{y}, y, t; \alpha_0 + \alpha_2, \alpha_1 + \alpha_2, -\alpha_2, \alpha_3 + \alpha_2), \\ s_3 : (x, y, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3) &\rightarrow (x, y - \frac{2\alpha_3}{x} + \frac{t}{x^2}, -t; \alpha_0, \alpha_1, \alpha_2 + 2\alpha_3, -\alpha_3), \\ \pi : (x, y, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3) &\rightarrow (\frac{x}{x-1}, -(x-1)((x-1)y + \alpha_2), -t; \alpha_1, \alpha_0, \alpha_2, \alpha_3). \end{aligned}$$

The list (5) should be read as

$$\begin{aligned} s_0(\alpha_0) &= -\alpha_0, & s_0(\alpha_1) &= \alpha_1, & s_0(\alpha_2) &= \alpha_2 + \alpha_0, & s_0(\alpha_3) &= \alpha_3, \\ s_0(x) &= x, & s_0(y) &= y - \frac{\alpha_0}{x-1}, & s_0(t) &= t. \end{aligned}$$

The above figure denotes the Dynkin diagram of type $B_3^{(1)}$. Let us set

$$f_0 := x - 1, \quad f_1 := x - \infty, \quad f_2 := y.$$

Following [5], we define the actions w_i ($i = 0, 1, 2$) as

$$(6) \quad w_i(g) = g + \frac{\alpha_i}{f_i} \{g, f_i\}, \quad g \in \mathbb{C}(t)[x, y],$$

where $\{, \}$ is the Poisson bracket such that $\{x, x\} = \{y, y\} = 0$, $\{x, y\} = 1$. These actions of w_i ($i = 0, 1, 2$) are equivalent to the actions of s_i ($i = 0, 1, 2$) given in Theorem 1.1. However, the actions of s_3 given in Theorem 1.1 are different from the actions defined by Noumi and Yamada in [5]. We also remark that $f := x$ is not an invariant divisor of the system (2).

In order to prove Theorem 1.1, we recall the definition of a symplectic transformation and its properties (see [3, 16]). Let

$$\varphi : x = x(X, Y, t), \quad y = y(X, Y, t), \quad t = t$$

be a biholomorphic mapping from a domain D in $\mathbb{C}^3 \ni (X, Y, t)$ into $\mathbb{C}^3 \ni (x, y, t)$. We say that the mapping is symplectic if

$$dx \wedge dy = dX \wedge dY,$$

where t is considered as a constant or a parameter, namely, if, for $t = t_0$, $\varphi_{t_0} = \varphi|_{t=t_0}$ is a symplectic mapping from the t_0 -section D_{t_0} of D to $\varphi(D_{t_0})$. Suppose that the mapping is symplectic. Then any Hamiltonian system

$$dx/dt = \partial H / \partial y, \quad dy/dt = -\partial H / \partial x$$

is transformed to

$$dX/dt = \partial K / \partial Y, \quad dY/dt = -\partial K / \partial X,$$

where

$$(A) \quad dx \wedge dy - dH \wedge dt = dX \wedge dY - dK \wedge dt.$$

Here t is considered as a variable. By this equation, the function K is determined by H uniquely modulo functions of t , namely, modulo functions independent of X and Y .

Proof of Theorem 1.1. At first, we consider the case of the transformation s_0 . Set

$$\begin{aligned} X &:= x, & Y &:= y - \frac{\alpha_0}{x-1}, & T &:= t, \\ A_0 &:= -\alpha_0, & A_1 &= \alpha_1, & A_2 &= \alpha_2 + \alpha_0, & A_3 &= \alpha_3. \end{aligned}$$

By resolving in $x, y, t, \alpha_0, \dots, \alpha_3$, we obtain

$$\begin{aligned} S_0 : \quad x &= X, & y &= Y - \frac{A_0}{X-1}, & t &= T, \\ \alpha_0 &= -A_0, & \alpha_1 &= A_1, & \alpha_2 &= A_2 + A_0, & \alpha_3 &= A_3. \end{aligned}$$

By S_0 , we obtain the polynomial Hamiltonian $S_0(H_1)$, and we see that

$$H_1 = (S_0(H_1) - A_0)|_{\{X \rightarrow x, Y \rightarrow y, T \rightarrow t, A_0 \rightarrow \alpha_0, A_1 \rightarrow \alpha_1, A_2 \rightarrow \alpha_2, A_3 \rightarrow \alpha_3\}}.$$

Since H_1 is modulo functions of t , we can check in the case of s_0 .

The cases of s_1, s_2 are similar. We note the relation between H_1 and the transformed Hamiltonian K_i ($i = 1, 2$), respectively: with the notation $res := \{X \rightarrow x, Y \rightarrow y, T \rightarrow t, A_0 \rightarrow \alpha_0, A_1 \rightarrow \alpha_1, A_2 \rightarrow \alpha_2, A_3 \rightarrow \alpha_3\}$

$$H_1 = K_1|_{res},$$

$$H_2 = \left(K_2 + \frac{A_2(A_2 + 2A_3 - 1 + T)}{T} \right)|_{res}.$$

Next, we consider the case of s_3 . Setting

$$S_3 : \quad \begin{aligned} x &= X, & y &= Y - \frac{2A_3}{X} + \frac{T}{X^2}, & t &= -T, \\ \alpha_0 &= A_0, & \alpha_1 &= A_1, & \alpha_2 &= A_2 + 2A_3, & \alpha_3 &= -A_3. \end{aligned}$$

Applying the transformation in t and the transformation of the symplectic 2-form:

$$dx \wedge dy = dX \wedge dY - d\left(\frac{1}{X}\right) \wedge dT,$$

we obtain the rational Hamiltonian $S_3(H_1)$, and we see that

$$H_1 = -\left\{ S_3(H_1) + \frac{1}{X} - (A_1 + 2A_2 + \frac{2A_3(T+1)}{T}) \right\}|_{res}.$$

Then we can check in the case of s_3 .

The case of π is similar. We note the relation between H_1 and the transformed Hamiltonian $\Pi(H_1)$ is given as follows:

$$H_1 = -\{\Pi(H_1) - A_2\}|_{res}.$$

This completes the proof. \square

Consider the following birational and symplectic transformations r_i (cf. [3, 16]):

$$(7) \quad \begin{aligned} r_0 : x_0 &= -((x-1)y - \alpha_0)y, & y_0 &= \frac{1}{y}, \\ r_1 : x_1 &= \frac{1}{x}, & y_1 &= -(yx + \alpha_1 + \alpha_2)x, \\ r_2 : x_2 &= \frac{1}{x}, & y_2 &= -(yx + \alpha_2)x, \\ r_3 : x_3 &= x, & y_3 &= y - \frac{2\alpha_3}{x} + \frac{t}{x^2}. \end{aligned}$$

These transformations are appeared as the patching data in the space of initial conditions of the system (2). The fact that the space of initial conditions of the system (2) is covered by this data will be cleared in the following paper.

Since each transformation r_i is symplectic, the system (2) is transformed into a Hamiltonian system, whose Hamiltonian may have poles. It is remarkable that the transformed system becomes again a polynomial system for any $i = 0, 1, 2, 3$. Furthermore, this holomorphy property uniquely characterizes the system (2).

Theorem 1.2. *Let us consider a polynomial Hamiltonian system with Hamiltonian $H \in \mathbb{C}(t)[x, y]$. We assume that*

(A1) *$\deg(H) = 5$ with respect to x, y .*

(A2) *This system becomes again a polynomial Hamiltonian system in each coordinate r_i ($i = 0, 1, 2, 3$).*

Then such a system coincides with the system (2).

We remark that if we look for a polynomial Hamiltonian system which admits the symmetry (5), we must consider cumbersome polynomials in variables x, y, t, α_i . On the other hand, in the holomorphy requirement (7), we only need to consider polynomials in x, y . This reduces the number of unknown coefficients drastically. On relations between s_i and r_i , see [15].

Proof of Theorem 1.2. At first, resolving the coordinate r_0 in the variables x, y , we obtain

$$(R_0) \quad x = -x_0 y_0^2 + \alpha_0 y_0 + 1, \quad y = \frac{1}{y_0}.$$

The polynomial H satisfying (A1) has 20 unknown coefficients in $\mathbb{C}(t)$. By R_0 , we transform H into $R_0(H)$, which has poles in only y_0 . For $R_0(H)$, we only have to determine the unknown coefficients so that they cancel the poles of $R_0(H)$.

In this way, we can obtain the Hamiltonian H_1 . □

Here we recall that the sixth Painlevé system is given by

$$(8) \quad \frac{dx}{dt} = \frac{\partial H_{VI}}{\partial y}, \quad \frac{dy}{dt} = -\frac{\partial H_{VI}}{\partial x}$$

with the polynomial Hamiltonian

$$(9) \quad \begin{aligned} & H_{VI}(x, y, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) \\ &= \frac{1}{t(t-1)} [y^2(x-t)(x-1)x - \{(\alpha_0-1)(x-1)x + \alpha_3(x-t)x \\ &+ \alpha_4(x-t)(x-1)\}y + \alpha_2(\alpha_1 + \alpha_2)x] \quad (\alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = 1). \end{aligned}$$

This system (8) admits affine Weyl group symmetry of type $D_4^{(1)}$ as the group of its Bäcklund transformations (see [6]), whose generators are explicitly given as follows:

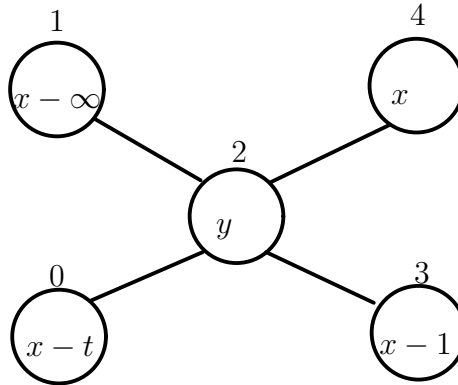


FIGURE 3. Dynkin diagram of type $D_4^{(1)}$

$$\begin{aligned}
w_0 : (x, y, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) &\rightarrow (x, y - \frac{\alpha_0}{x-t}, t; -\alpha_0, \alpha_1, \alpha_2 + \alpha_0, \alpha_3, \alpha_4), \\
w_1 : (x, y, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) &\rightarrow (x, y, t; \alpha_0, -\alpha_1, \alpha_2 + \alpha_1, \alpha_3, \alpha_4), \\
w_2 : (x, y, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) &\rightarrow (x + \frac{\alpha_2}{y}, y, t; \\
&\quad \alpha_0 + \alpha_2, \alpha_1 + \alpha_2, -\alpha_2, \alpha_3 + \alpha_2, \alpha_4 + \alpha_2), \\
w_3 : (x, y, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) &\rightarrow (x, y - \frac{\alpha_3}{x-1}, t; \alpha_0, \alpha_1, \alpha_2 + \alpha_3, -\alpha_3, \alpha_4), \\
w_4 : (x, y, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) &\rightarrow (x, y - \frac{\alpha_4}{x}, t; \alpha_0, \alpha_1, \alpha_2 + \alpha_4, \alpha_3, -\alpha_4).
\end{aligned}$$

In addition to Theorems 1.1 and 1.2, we give an explicit description of a confluence from the sixth Painlevé system (8) to the system (2).

Theorem 1.3. *For the sixth Painlevé system (8), we make the change of parameters and variables*

$$(10) \quad \alpha_0 = \varepsilon^{-1}, \quad \alpha_1 = A_0, \quad \alpha_2 = A_2, \quad \alpha_3 = \frac{2A_3\varepsilon - 1}{\varepsilon}, \quad \alpha_4 = A_1,$$

$$(11) \quad t = 1 - \varepsilon T, \quad x = \frac{1}{1 - X}, \quad y = X^2Y - 2XY + A_2X + Y - A_2$$

from $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, t, x, y$ to $A_0, A_1, A_2, A_3, \varepsilon, T, X, Y$. Then the system (8) can also be written in the new variables T, X, Y and parameters $A_0, A_1, A_2, A_3, \varepsilon$ as a Hamiltonian system. This new system tends to the system (2) as $\varepsilon \rightarrow 0$.

By proving the following theorem, we see how the degeneration process given in Theorem 1.3 works on the Bäcklund transformation group $W(D_4^{(1)})$ (cf. [17]).

Theorem 1.4. *For the degeneration process in Theorem 1.3, we can choose a subgroup $W_{D_4^{(1)} \rightarrow B_3^{(1)}}$ of the Bäcklund transformation group $W(D_4^{(1)})$ so that $W_{D_4^{(1)} \rightarrow B_3^{(1)}}$ converges to the Bäcklund transformation group $W(B_3^{(1)})$ of the system (2) as $\varepsilon \rightarrow 0$.*

Proof of Theorem 1.4. Notice that

$$A_0 + A_1 + 2A_2 + 2A_3 = \alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = 1$$

and the change of variables from (x, y) to (X, Y) is symplectic.

Let us see the actions of the generators w_i , $i = 0, 1, 2, 3, 4$ on the parameters A_i , $i = 0, 1, 2, 3$ and ε where

$$A_0 = \alpha_1, \quad A_1 = \alpha_4, \quad A_2 = \alpha_2, \quad A_3 = \frac{\alpha_0 + \alpha_3}{2}, \quad \varepsilon = \frac{1}{\alpha_0}.$$

By a direct calculation, we have

$$\begin{aligned}
w_0(A_0, A_1, A_2, A_3, \varepsilon) &\longrightarrow (A_0, A_1, A_2 + \frac{1}{\varepsilon}, A_3 - \frac{1}{\varepsilon}, -\varepsilon), \\
w_1(A_0, A_1, A_2, A_3, \varepsilon) &\longrightarrow (-A_0, A_1, A_2 + A_0, A_3, \varepsilon), \\
w_2(A_0, A_1, A_2, A_3, \varepsilon) &\longrightarrow (A_0 + A_2, A_1 + A_2, -A_2, A_3 + A_2, \frac{\varepsilon}{1 + \varepsilon A_2}), \\
w_3(A_0, A_1, A_2, A_3, \varepsilon) &\longrightarrow (A_0, A_1, A_2 + 2A_3 - \frac{1}{\varepsilon}, -A_3 + \frac{1}{\varepsilon}, \varepsilon), \\
w_4(A_0, A_1, A_2, A_3, \varepsilon) &\longrightarrow (A_0, -A_1, A_2 + A_1, A_3, \varepsilon).
\end{aligned}$$

We remark that $w_0(A_2), w_0(A_3), w_3(A_2)$ and $w_3(A_3)$ diverge as $\varepsilon \rightarrow 0$.

Observing these relations, we take a subgroup $W_{D_4^{(1)} \rightarrow B_3^{(1)}}$ of $W(D_4^{(1)})$ generated by S_0, S_1, S_2, S_3 defined by

$$S_0 := w_1, \quad S_1 := w_4, \quad S_2 := w_2, \quad S_3 := w_0 w_3.$$

We can easily check

(12)

$$\begin{aligned} S_0(A_0, A_1, A_2, A_3, \varepsilon) &\longrightarrow (-A_0, A_1, A_2 + A_0, A_3, \varepsilon), \\ S_1(A_0, A_1, A_2, A_3, \varepsilon) &\longrightarrow (A_0, -A_1, A_2 + A_1, A_3, \varepsilon), \\ S_2(A_0, A_1, A_2, A_3, \varepsilon) &\longrightarrow (A_0 + A_2, A_1 + A_2, -A_2, A_3 + A_2, \frac{\varepsilon}{1 + \varepsilon A_2}), \\ S_3(A_0, A_1, A_2, A_3, \varepsilon) &\longrightarrow (A_0, A_1, A_2 + 2A_3, -A_3, -\varepsilon), \end{aligned}$$

and the generators satisfy the following relations:

$$(S_i)^2 = 1, \quad (S_0 S_1)^2 = (S_0 S_3)^2 = (S_1 S_3)^2 = 1, \quad (S_0 S_2)^3 = (S_1 S_2)^3 = 1, \quad (S_2 S_3)^4 = 1.$$

In short, the group $W_{D_4^{(1)} \rightarrow B_3^{(1)}} = \langle S_0, S_1, S_2, S_3 \rangle$ can be considered to be an affine Weyl group of the affine Lie algebra of type $B_3^{(1)}$ with simple roots A_0, A_1, A_2, A_3 .

Now we investigate how the generators of $W_{D_4^{(1)} \rightarrow B_3^{(1)}}$ act on T, X and Y . We can verify

(13)

$$\begin{aligned} S_0(X, Y, T) &\longrightarrow (X, Y - \frac{A_0}{X-1}, T), \\ S_1(X, Y, T) &\longrightarrow (X, Y, T), \\ S_2(X, Y, T) &\longrightarrow (X + \frac{A_2}{Y}, Y, T(1 + \varepsilon A_2)), \\ S_3(X, Y, T) &\longrightarrow (X, \frac{(\varepsilon T - 1)X^2 Y - \varepsilon T X Y - 2A_3(\varepsilon T - 1)X + (2\varepsilon A_3 - 1)T}{\{(\varepsilon T - 1)X - \varepsilon T\}X}, -T). \end{aligned}$$

Here $S_3(T) = -T$ can be understood as follows: by using the relation $T = \alpha_0(1 - t)$, the action of S_3 on T is obtained as

$$\begin{aligned} S_3(T) &= w_0 \circ w_3(\alpha_0(1 - t)) \\ &= w_0(\alpha_0(1 - t)) \\ &= -\alpha_0(1 - t) \\ &= -T. \end{aligned}$$

By comparing (12), (13) with s_i ($i = 0, 1, 2, 3$) given in Theorem 1.1, we see that our theorem holds. \square

By the following theorem, we will show that the system (2) coincides with the system of type $A_3^{(1)}$ (see [4, 5, 6]).

Theorem 1.5. *For the system (2), we make the change of parameters and variables*

$$(14) \quad \beta_0 = \alpha_2 + 2\alpha_3, \quad \beta_1 = \alpha_1, \quad \beta_2 = \alpha_2, \quad \beta_3 = \alpha_0,$$

$$(15) \quad X = \frac{1}{x}, \quad Y = -x(xy + \alpha_2), \quad T = -t$$

from $\alpha_0, \alpha_1, \alpha_2, \alpha_3, t, x, y$ to $\beta_0, \beta_1, \beta_2, \beta_3, T, X, Y$. Then the system (2) can also be written in the new variables T, X, Y and parameters $\beta_0, \beta_1, \beta_2, \beta_3$ as a Hamiltonian system. This new system tends to the system $A_3^{(1)}$:

$$(16) \quad \begin{cases} \frac{dX}{dT} = \frac{2X^2Y}{T} + X^2 - \frac{2XY}{T} - (1 + \frac{\beta_1 + \beta_3}{T})X + \frac{\beta_1}{T}, \\ \frac{dY}{dT} = -\frac{2XY^2}{T} + \frac{Y^2}{T} - 2XY + (1 + \frac{\beta_1 + \beta_3}{T})Y - \beta_2 \end{cases}$$

with the Hamiltonian

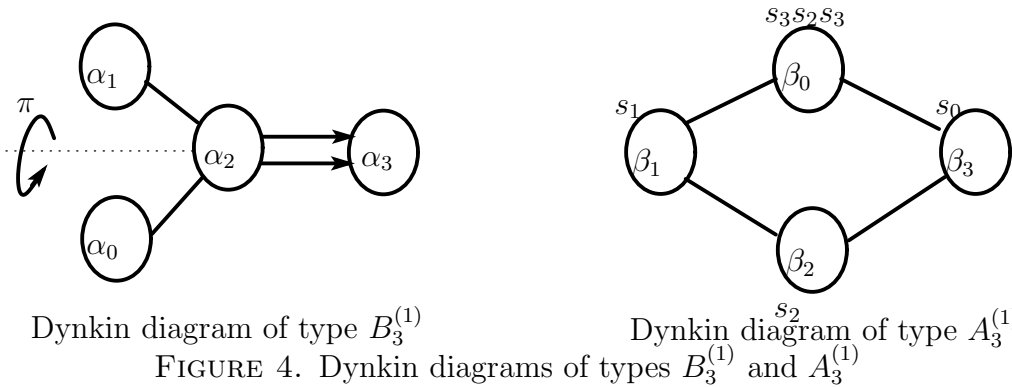
$$(17) \quad \begin{aligned} & H_{A_3^{(1)}}(X, Y, T; \beta_1, \beta_2, \beta_3) \\ &= \frac{Y(Y+T)X(X-1) + \beta_2XT - \beta_3XY - \beta_1Y(X-1)}{T}. \end{aligned}$$

By putting $q = 1 - \frac{1}{X}$, we have the Painlevé V equation (see [20]):

$$\frac{d^2q}{dT^2} = \left(\frac{1}{2q} + \frac{1}{q-1} \right) \left(\frac{dq}{dT} \right)^2 - \frac{1}{T} \frac{dq}{dT} + \frac{(q-1)^2}{T^2} \left(aq + \frac{b}{q} \right) + c \frac{q}{T} + d \frac{q(q+1)}{q-1},$$

where

$$a = \frac{\beta_1^2}{2}, \quad b = -\frac{\beta_3^2}{2}, \quad c = \beta_0 - \beta_2, \quad d = -\frac{1}{2}.$$



Proof of Theorem 1.5. Notice that

$$\alpha_0 + \alpha_1 + 2\alpha_2 + 2\alpha_3 = \beta_0 + \beta_1 + \beta_2 + \beta_3 = 1$$

and the change of variables from (x, y, t) to (X, Y, T) in Theorem 1.5 is symplectic. Choose S_i as

$$S_0 := s_3 s_2 s_3, \quad S_1 := s_1, \quad S_2 := s_2, \quad S_3 := s_0, \quad S_4 := s_3, \quad S_5 := \pi.$$

The transformations S_0, S_1, S_2, S_3 are reflections of

$$\beta_0 = \alpha_2 + 2\alpha_3, \quad \beta_1 = \alpha_1, \quad \beta_2 = \alpha_2, \quad \beta_3 = \alpha_0, \text{ respectively.}$$

We can verify

$$\begin{aligned}
S_0 &: (X, Y, T; \beta_0, \beta_1, \beta_2, \beta_3) \rightarrow (X + \frac{\beta_0}{Y+T}, Y, T; -\beta_0, \beta_1 + \beta_0, \beta_2, \beta_3 + \beta_0), \\
S_1 &: (X, Y, T; \beta_0, \beta_1, \beta_2, \beta_3) \rightarrow (X, Y - \frac{\beta_1}{X}, T; \beta_0 + \beta_1, -\beta_1, \beta_2 + \beta_1, \beta_3), \\
S_2 &: (X, Y, T; \beta_0, \beta_1, \beta_2, \beta_3) \rightarrow (X + \frac{\beta_2}{Y}, Y, T; \beta_0, \beta_1 + \beta_2, -\beta_2, \beta_3 + \beta_2), \\
S_3 &: (X, Y, T; \beta_0, \beta_1, \beta_2, \beta_3) \rightarrow (X, Y - \frac{\beta_3}{X-1}, T; \beta_0 + \beta_3, \beta_1, \beta_2 + \beta_3, -\beta_3), \\
S_4 &: (X, Y, T; \beta_0, \beta_1, \beta_2, \beta_3) \rightarrow (X, Y + T, -T; \beta_2, \beta_1, \beta_0, \beta_3), \\
S_5 &: (X, Y, T; \beta_0, \beta_1, \beta_2, \beta_3) \rightarrow (1 - X, -Y, -T; \beta_0, \beta_3, \beta_2, \beta_1).
\end{aligned}$$

The proof has thus been completed. \square

Proposition 1.1. *The system (2) admits the following transformation φ as its Bäcklund transformation:*

$$\begin{aligned}
&\varphi : (x, y, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3) \\
&\rightarrow \left(-\frac{t}{x(xy + \alpha_2)}, -\frac{x(xy + \alpha_2)(-xy + x^2y - \alpha_0 - \alpha_2 + \alpha_2x)}{t}, t; \right. \\
&\quad \left. \alpha_2 + 2\alpha_3, \alpha_2, \alpha_0, \frac{\alpha_1 - \alpha_0}{2} \right).
\end{aligned}$$

We note that this transformation φ is pulled back the diagram automorphism π of the system (16)

$$\pi : (X, Y, T; \beta_0, \beta_1, \beta_2, \beta_3) \rightarrow \left(-\frac{Y}{T}, (X-1)T, T; \beta_1, \beta_2, \beta_3, \beta_0 \right)$$

by transformations (14) and (15).

2. THE SYSTEM OF TYPE $G_2^{(1)}$

In this section, we present a 2-parameter family of two-dimensional polynomial Hamiltonian systems given by

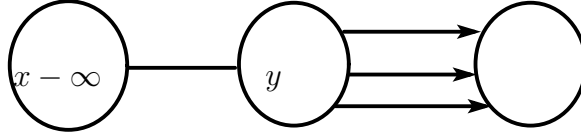
$$(18) \quad \begin{cases} \frac{dx}{dt} = 4x^3y + 2(\alpha_0 + 2\alpha_1)x^2 + 2tx + 1, \\ \frac{dy}{dt} = -6x^2y^2 - 4(\alpha_0 + 2\alpha_1)xy - 2ty - 2\alpha_1(\alpha_0 + \alpha_1) \end{cases}$$

with the polynomial Hamiltonian

$$\begin{aligned}
(19) \quad &H_{G_2^{(1)}}(x, y, t; \alpha_0, \alpha_1, \alpha_2) \\
&= 2x^3y^2 + 2(\alpha_0 + 2\alpha_1)x^2y + 2txy + 2\alpha_1(\alpha_0 + \alpha_1)x + y.
\end{aligned}$$

Here x and y denote unknown complex variables and $\alpha_0, \alpha_1, \alpha_2$ are complex parameters satisfying the relation:

$$(20) \quad \alpha_0 + 2\alpha_1 + 3\alpha_2 = 1.$$

FIGURE 5. Dynkin diagram of type $G_2^{(1)}$

Theorem 2.1. *The system (18) admits extended affine Weyl group symmetry of type $G_2^{(1)}$ as the group of its Bäcklund transformations (cf. [6]), whose generators are explicitly given as follows:*

$$\begin{aligned} s_0 &: (x, y, t; \alpha_0, \alpha_1, \alpha_2) \rightarrow (x, y, t; -\alpha_0, \alpha_1 + \alpha_0, \alpha_2), \\ s_1 &: (x, y, t; \alpha_0, \alpha_1, \alpha_2) \rightarrow (x + \frac{\alpha_1}{y}, y, t; \alpha_0 + \alpha_1, -\alpha_1, \alpha_2 + \alpha_1), \\ s_2 &: (x, y, t; \alpha_0, \alpha_1, \alpha_2) \rightarrow (\sqrt{-1}x, -\sqrt{-1}(y - \frac{3\alpha_2}{x} + \frac{t}{x^2} + \frac{1}{2x^3}), -\sqrt{-1}t; \\ &\quad \alpha_0, \alpha_1 + 3\alpha_2, -\alpha_2). \end{aligned}$$

Theorem 2.2. *Let us consider a polynomial Hamiltonian system with Hamiltonian $H \in \mathbb{C}(t)[x, y]$. We assume that*

(A1) *$\deg(H) = 5$ with respect to x, y .*

(A2) *This system becomes again a polynomial Hamiltonian system in each coordinate r_i ($i = 0, 1, 2$) (cf. [3]):*

$$\begin{aligned} r_0 &: x_0 = \frac{1}{x}, \quad y_0 = -(yx + \alpha_0 + \alpha_1)x, \\ r_1 &: x_1 = \frac{1}{x}, \quad y_1 = -(yx + \alpha_1)x, \\ r_2 &: x_2 = x, \quad y_2 = y - \frac{3\alpha_2}{x} + \frac{t}{x^2} + \frac{1}{2x^3}. \end{aligned}$$

Then such a system coincides with the system (18).

Theorems 2.1, 2.2 can be checked by a direct calculation, respectively.

Theorem 2.3. *For the system (2), we make the change of parameters and variables*

$$(21) \quad \alpha_0 = -\frac{1}{2\varepsilon^2}, \quad \alpha_1 = A_0, \quad \alpha_2 = A_1, \quad \alpha_3 = \frac{1 + 6A_2\varepsilon^2}{4\varepsilon^2},$$

$$(22) \quad t = \frac{-1 - 2\varepsilon T}{2\varepsilon^2}, \quad x = \frac{\varepsilon - X}{\varepsilon}, \quad y = -\varepsilon Y$$

from $\alpha_0, \alpha_1, \alpha_2, \alpha_3, t, x, y$ to $A_0, A_1, A_2, \varepsilon, T, X, Y$. Then the system (2) can also be written in the new variables T, X, Y and parameters $A_0, A_1, A_2, \varepsilon$ as a Hamiltonian system. This new system tends to the system (18) as $\varepsilon \rightarrow 0$.

By proving the following theorem, we see how the degeneration process given in Theorem 2.3 works on the Bäcklund transformation group $W(B_3^{(1)})$ (cf. [17]).

Theorem 2.4. *For the degeneration process in Theorem 2.3, we can choose a subgroup $W_{B_3^{(1)} \rightarrow G_2^{(1)}}$ of the Bäcklund transformation group $W(B_3^{(1)})$ so that $W_{B_3^{(1)} \rightarrow G_2^{(1)}}$ converges to the Bäcklund transformation group $W(G_2^{(1)})$ of the system (18) as $\varepsilon \rightarrow 0$.*

Proof of Theorem 2.4. Notice that

$$A_0 + 2A_1 + 3A_2 = \alpha_0 + \alpha_1 + 2\alpha_2 + 2\alpha_3 = 1$$

and the change of variables from (x, y) to (X, Y) is symplectic, however the change of parameters (21) is not one to one differently from the case of $P_{VI} \rightarrow P_V$.

Choose S_i ($i = 0, 1, 2$) as

$$S_0 := s_1, \quad S_1 := s_2, \quad S_2 := s_0 s_3,$$

and set $W_{B_3^{(1)} \rightarrow G_2^{(1)}} = \langle S_0, S_1, S_2 \rangle$. Then we immediately have

$$\begin{aligned} S_0(A_0, A_1, A_2) &= (-A_0, A_1 + A_0, A_2), \\ S_1(A_0, A_1, A_2) &= (A_0 + A_1, -A_1, A_2 + A_1), \\ S_2(A_0, A_1, A_2) &= (A_0, A_1 + 3A_2, -A_2). \end{aligned}$$

However, we see that $S_i(\varepsilon)$ have ambiguities of signature. For example, since

$$S_2(\varepsilon^2) = s_0 \circ s_3 \left(-\frac{1}{2\alpha_0} \right) = -\frac{1}{2} s_0 \left(\frac{1}{\alpha_0} \right) = -\frac{1}{2} \left(-\frac{1}{\alpha_0} \right) = \frac{1}{2\alpha_0} = -\varepsilon^2,$$

we can choose any one of the two branches as $S_2(\varepsilon)$. Among such possibilities, we take a choice as

$$(23) \quad S_0(\varepsilon) = \varepsilon, \quad S_1(\varepsilon) = \varepsilon(1 - 2A_1\varepsilon^2)^{-1/2}, \quad S_2(\varepsilon) = \sqrt{-1}\varepsilon,$$

where $(1 - 2A_1\varepsilon^2)^{-1/2} = 1$ at $A_1\varepsilon^2 = 0$, or considering in the category of formal power series, we make a convention that $(1 - 2A_1\varepsilon^2)^{-1/2}$ is formal power series of $A_1\varepsilon^2$ with constant term 1 according to

$$(1 + x)^c \sim 1 + \sum_{n \geq 1} \binom{c}{n} x^n.$$

We notice that the generators acting on parameters $A_0, A_1, A_2, \varepsilon$ satisfy the following relations:

$$S_i^2 = 1, \quad (S_0 S_2)^2 = 1, \quad (S_0 S_1)^3 = 1, \quad (S_1 S_2)^6 = 1.$$

Now we observe the actions of S_i , $i = 0, 1, 2$ on the variables X, Y, T . By means of (22), (23) and

$$S_0(t) = s_1(t) = t, \quad S_1(t) = s_2(t) = t, \quad S_2(t) = s_0 \circ s_3(t) = -t,$$

we can easily check

$$S_0(T) = T, \quad S_1(T) = (T + A_1\varepsilon)(1 - 2A_1\varepsilon^2)^{-1/2}, \quad S_2(T) = -\sqrt{-1}T.$$

By (21), (22), (23) and the actions of s_1, s_2 on x, y , we can easily verify

$$\begin{aligned} S_0(X, Y) &= (X, Y), \\ S_1(X, Y) &= \left(X + \frac{A_1}{Y}, Y \right). \end{aligned}$$

The form of the actions $S_2 = s_0 s_3$ on X and Y are complicated, but we can see that

$$S_2(X, Y) = \left(\sqrt{-1}X, -\sqrt{-1}\left(Y - \frac{3A_2}{X} + \frac{T}{X^2} + \frac{1}{2X^3}\right) \right).$$

The proof has thus been completed. \square

By the following theorem, we will show that the system (18) coincides with the system of type $A_2^{(1)}$ (see [4, 5, 6]).

Theorem 2.5. *For the system (18), we make the change of parameters and variables*

$$(24) \quad \beta_0 = \alpha_1 + 3\alpha_2, \quad \beta_1 = \alpha_0, \quad \beta_2 = \alpha_1,$$

$$(25) \quad X = \frac{1}{x}, \quad Y = -x(xy + \alpha_1), \quad T = t$$

from $\alpha_0, \alpha_1, \alpha_2, t, x, y$ to $\beta_0, \beta_1, \beta_2, T, X, Y$. Then the system (18) can also be written in the new variables T, X, Y and parameters $\beta_0, \beta_1, \beta_2$ as a Hamiltonian system. This new system tends to the system $A_2^{(1)}$:

$$(26) \quad \begin{cases} \frac{dX}{dT} = -X^2 + 4XY - 2TX - 2\beta_1, \\ \frac{dY}{dT} = -2Y^2 + 2XY + 2TY + \beta_2 \end{cases}$$

with the Hamiltonian

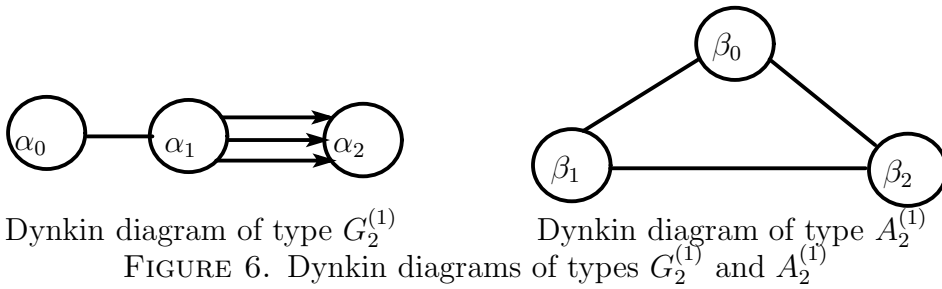
$$(27) \quad \begin{aligned} H_{A_2^{(1)}}(X, Y, T; \beta_0, \beta_1, \beta_2) \\ = -2TXY - X^2Y + 2XY^2 - 2\beta_1Y - \beta_2X. \end{aligned}$$

By putting $q = X$, we have the Painlevé IV equation:

$$\frac{d^2q}{dT^2} = \frac{1}{2q} \left(\frac{dq}{dT} \right)^2 + \frac{3}{2}q^3 + 4Tq^2 + 2(T^2 - a)q + \frac{b}{q},$$

where

$$a = 1 - \beta_1 - 2\beta_2, \quad b = -2\beta_1^2.$$



Proof of Theorem 2.5. Notice that

$$\alpha_0 + 2\alpha_1 + 3\alpha_2 = \beta_0 + \beta_1 + \beta_2 = 1$$

and the change of variables from (x, y, t) to (X, Y, T) in Theorem 2.5 is symplectic. Choose S_i as

$$S_0 := s_2 s_1 s_2, \quad S_1 := s_0, \quad S_2 := s_1, \quad S_3 := s_2.$$

The transformations S_0, S_1, S_2 are reflections of

$$\beta_0 = \alpha_1 + 3\alpha_2, \quad \beta_1 = \alpha_0, \quad \beta_2 = \alpha_1 \text{ respectively.}$$

We can verify

$$\begin{aligned}
S_0 : (X, Y, T; \beta_0, \beta_1, \beta_2) &\rightarrow (X + \frac{2\beta_0}{2Y - X - 2T}, Y + \frac{\beta_0}{2Y - X - 2T}, T; \\
&\quad -\beta_0, \beta_1 + \beta_0, \beta_2 + \beta_0), \\
S_1 : (X, Y, T; \beta_0, \beta_1, \beta_2) &\rightarrow (X, Y - \frac{\beta_1}{X}, T; \beta_0 + \beta_1, -\beta_1, \beta_2 + \beta_1), \\
S_2 : (X, Y, T; \beta_0, \beta_1, \beta_2) &\rightarrow (X + \frac{\beta_2}{Y}, Y, T; \beta_0 + \beta_2, \beta_1 + \beta_2, -\beta_2), \\
S_3 : (X, Y, T; \beta_0, \beta_1, \beta_2) &\rightarrow (-\sqrt{-1}X, -\frac{\sqrt{-1}(X - 2Y + 2T)}{2}, -\sqrt{-1}T; \beta_2, \beta_1, \beta_0).
\end{aligned}$$

The proof has thus been completed. \square

Proposition 2.1. *The system (18) admits the following transformation φ as its Bäcklund transformation:*

$$\begin{aligned}
\varphi : (x, y, t; \alpha_0, \alpha_1, \alpha_2) &\rightarrow (\frac{1}{2x(xy + \alpha_1)}, \\
&\quad -2x(xy + \alpha_1)\{xy + 2tx^2y + 2x^4y^2 + 4\alpha_1x^3y + 2\alpha_1^2x^2 + 2\alpha_1tx + 2\alpha_1 + 3\alpha_2\}, t; \\
&\quad \alpha_1, \alpha_1 + 3\alpha_2, \frac{\alpha_0 - \alpha_1 - 3\alpha_2}{3}).
\end{aligned}$$

We note that this transformation φ is pulled back the diagram automorphism π of the system (26) by the symplectic transformation defined in Theorem 2.5

$$\pi : (X, Y, T; \beta_0, \beta_1, \beta_2) \rightarrow (-2Y, \frac{2T + X - 2Y}{2}, T; \beta_1, \beta_2, \beta_0).$$

3. THE SYSTEM OF $D_3^{(2)}$

In this section, we present a 2-parameter family of polynomial Hamiltonian systems given by

$$(28) \quad \begin{cases} \frac{dx}{dt} = \frac{2x^2y}{t} - x^2 - \frac{2\alpha_0x}{t} + \frac{1}{t}, \\ \frac{dy}{dt} = -\frac{2xy^2}{t} + 2xy + \frac{2\alpha_0y}{t} + \alpha_1 \end{cases}$$

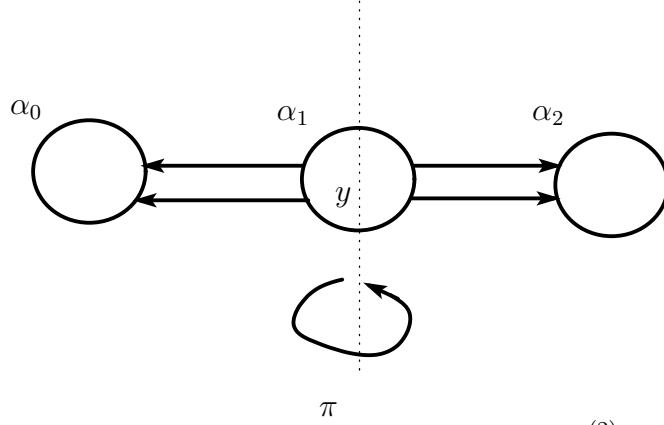
with the polynomial Hamiltonian

$$(29) \quad H_{D_3^{(2)}}(x, y, t; \alpha_0, \alpha_1, \alpha_2) = \frac{x^2y(y - t) - (2\alpha_0y + t\alpha_1)x + y}{t}.$$

Here x and y denote unknown complex variables and $\alpha_0, \alpha_1, \alpha_2$ are complex parameters satisfying the relation:

$$(30) \quad \alpha_0 + \alpha_1 + \alpha_2 = \frac{1}{2}.$$

Theorem 3.1. *The system (28) admits extended affine Weyl group symmetry of type $D_3^{(2)}$ as the group of its Bäcklund transformations (cf. [6]), whose generators are explicitly given as follows:*

FIGURE 7. Dynkin diagram of type $D_3^{(2)}$

$$\begin{aligned}
s_0 : (x, y, t; \alpha_0, \alpha_1, \alpha_2) &\rightarrow (-x, -y + \frac{2\alpha_0}{x} - \frac{1}{x^2}, -t; -\alpha_0, \alpha_1 + 2\alpha_0, \alpha_2), \\
s_1 : (x, y, t; \alpha_0, \alpha_1, \alpha_2) &\rightarrow (x + \frac{\alpha_1}{y}, y, t; \alpha_0 + \alpha_1, -\alpha_1, \alpha_2 + \alpha_1), \\
s_2 : (x, y, t; \alpha_0, \alpha_1, \alpha_2) &\rightarrow (x, y - t, -t; \alpha_0, \alpha_1 + 2\alpha_2, -\alpha_2), \\
\pi : (x, y, t; \alpha_0, \alpha_1, \alpha_2) &\rightarrow (\frac{1}{tx}, -tx(yx + \alpha_1), t; \alpha_2, \alpha_1, \alpha_0).
\end{aligned}$$

Theorem 3.2. *Let us consider a polynomial Hamiltonian system with Hamiltonian $H \in \mathbb{C}(t)[x, y]$. We assume that*

(A1) $\deg(H) = 5$ with respect to x, y .

(A2) *This system becomes again a polynomial Hamiltonian system in each coordinate r_i ($i = 0, 1, 2$) (cf. [3]):*

$$\begin{aligned}
r_0 : x_0 &= x, \quad y_0 = y - \frac{2\alpha_0}{x} + \frac{1}{x^2}, \\
r_1 : x_1 &= \frac{1}{x}, \quad y_1 = -(yx + \alpha_1)x, \\
r_2 : x_2 &= \frac{1}{x}, \quad y_2 = -((y - t)x + \alpha_1 + 2\alpha_2)x.
\end{aligned}$$

Then such a system coincides with the system (28).

Theorem 3.3. *For the system (2), we make the change of parameters and variables:*

$$(31) \quad x = \frac{\varepsilon TX}{1 + \varepsilon TX}, \quad y = \frac{(1 + \varepsilon TX)(\varepsilon TXY + Y + A_1 \varepsilon T)}{\varepsilon T}, \quad t = \varepsilon T,$$

$$(32) \quad \alpha_0 = -\frac{1 - 2A_2 \varepsilon}{\varepsilon}, \quad \alpha_1 = \frac{1}{\varepsilon}, \quad \alpha_2 = A_1, \quad \alpha_3 = A_0$$

from $x, y, t, \alpha_0, \alpha_1, \alpha_2, \alpha_3$ to $X, Y, T, A_0, A_1, A_2, \varepsilon$. Then the system (2) can also be written in the new variables T, X, Y and parameters $A_0, A_1, A_2, \varepsilon$ as a Hamiltonian system. This new system tends to the system (28) as $\varepsilon \rightarrow 0$.

By proving the following theorem, we see how the degeneration process given in Theorem 3.3 works on the Bäcklund transformation group $W(B_3^{(1)})$ (cf. [17]).

Theorem 3.4. *For the degeneration process in Theorem 3.3, we can choose a subgroup $W_{B_3^{(1)} \rightarrow D_3^{(2)}}$ of the Bäcklund transformation group $W(B_3^{(1)})$ so that $W_{B_3^{(1)} \rightarrow D_3^{(2)}}$*

converges to the Bäcklund transformation group $W(D_3^{(2)})$ of the system (28) as $\varepsilon \rightarrow 0$.

Proof of Theorem 3.4. Notice that

$$2(A_0 + A_1 + A_2) = \alpha_0 + \alpha_1 + 2\alpha_2 + 2\alpha_3 = 1$$

and the change of variables from (x, y) to (X, Y) is symplectic. Choose S_i ($i = 0, 1, 2$) as

$$S_0 := s_3, \quad S_1 := s_2, \quad S_2 := s_0 s_1.$$

Then the transformations S_i are reflections of the parameters A_0, A_1, A_2 . The transformation group $\langle S_0, S_1, S_2 \rangle$ coincides with the transformations given in Theorem 3.1. The proof has thus been completed. \square

By the following theorem, we will show that the system (28) coincides with the system of type $C_2^{(1)}$ (see [6, 20]).

Theorem 3.5. *For the system (28), we make the change of parameters and variables:*

$$(33) \quad \alpha_0 = \frac{\beta_2 - \beta_0}{2}, \quad \alpha_1 = \beta_0, \quad \alpha_2 = \beta_1,$$

$$(34) \quad x = \frac{1}{X}, \quad y = -X(YX + \beta_0), \quad T = t$$

from $x, y, t, \alpha_0, \alpha_1, \alpha_2$ to $X, Y, T, \beta_0, \beta_1, \beta_2$. Then this new system coincides with the system of type $C_2^{(1)}$ (see [6]):

$$(35) \quad \begin{cases} \frac{dX}{dT} = \frac{2X^2Y}{T} - \frac{X^2}{T} + \frac{(\beta_0 + \beta_2)X}{T} + 1, \\ \frac{dY}{dT} = -\frac{2XY^2}{T} + \frac{2XY}{T} - \frac{(\beta_0 + \beta_2)Y}{T} + \frac{\beta_0}{T} \end{cases}$$

with the Hamiltonian

$$(36) \quad H_{C_2^{(1)}}(X, Y, T; \beta_0, \beta_1, \beta_2) = \frac{X^2Y(Y-1) + X[(\beta_0 + \beta_2)Y - \beta_0] + TY}{T} \\ (\beta_0 + 2\beta_1 + \beta_2 = 1).$$

By putting $q = \frac{x}{\tau}$, $T = \tau^2$, we will see that this system (35) is equivalent to the third Painlevé equation:

$$\frac{d^2q}{d\tau^2} = \frac{1}{q} \left(\frac{dq}{d\tau} \right)^2 - \frac{1}{\tau} \frac{dq}{d\tau} + \frac{1}{\tau} (aq^2 + b) + cq^3 + \frac{d}{q},$$

where

$$a = 4(\beta_0 - \beta_2), \quad b = -4(\beta_0 + \beta_2 - 1), \quad c = 4, \quad d = -4.$$

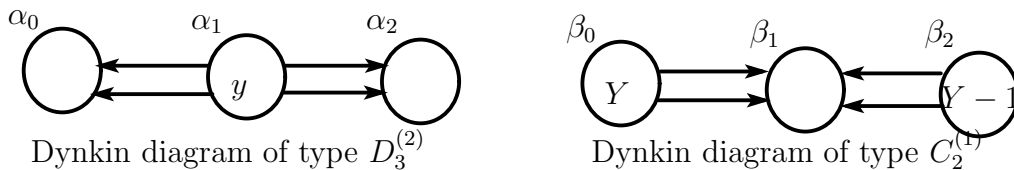


FIGURE 8. Dynkin diagrams of types $D_3^{(2)}$ and $C_2^{(1)}$

Proof of Theorem 3.5. Notice that

$$2(\alpha_0 + \alpha_1 + \alpha_2) = \beta_0 + 2\beta_1 + \beta_2 = 1$$

and the change of variables from (x, y, t) to (X, Y, T) in Theorem 3.5 is symplectic. Choose S_i , $i = 0, 1, 2, 3$ as

$$S_0 := s_1, \quad S_1 := s_2, \quad S_2 := s_0 s_1 s_0, \quad S_3 := s_0.$$

The transformations S_0, S_1, S_2 are reflections of

$$\beta_0 = \alpha_1, \quad \beta_1 = \alpha_2, \quad \beta_2 = 2\alpha_0 + \alpha_1 \text{ respectively.}$$

We can verify

$$\begin{aligned} S_0 : (X, Y, T; \beta_0, \beta_1, \beta_2) &\rightarrow (X + \frac{\beta_0}{Y}, Y, T; -\beta_0, \beta_1 + \beta_0, \beta_2), \\ S_1 : (X, Y, T; \beta_0, \beta_1, \beta_2) &\rightarrow (X, Y - \frac{2\beta_1}{X} + \frac{T}{X^2}, -T; \beta_0 + 2\beta_1, -\beta_1, \beta_2 + 2\beta_1), \\ S_2 : (X, Y, T; \beta_0, \beta_1, \beta_2) &\rightarrow (X + \frac{\beta_2}{Y-1}, Y, T; \beta_0, \beta_1 + \beta_2, -\beta_2), \\ S_3 : (X, Y, T; \beta_0, \beta_1, \beta_2) &\rightarrow (-X, 1-Y, -T; \beta_2, \beta_1, \beta_0). \end{aligned}$$

The proof has thus been completed. \square

4. THE SYSTEM OF $A_2^{(2)}$

In this section, we present a 1-parameter family of polynomial Hamiltonian systems given by

$$(37) \quad \begin{cases} \frac{dx}{dt} = x^4 y + \alpha_0 x^3 + \frac{t}{2} x^2 + 1, \\ \frac{dy}{dt} = -2x^3 y^2 - 3\alpha_0 x^2 y - txy - \alpha_0^2 x - \frac{\alpha_0 t}{2} \end{cases}$$

with the polynomial Hamiltonian

$$(38) \quad H_{A_2^{(2)}}(x, y, t; \alpha_0, \alpha_1) = \frac{x^4 y^2}{2} + \alpha_0 x^3 y + \frac{1}{2} t x^2 y + \frac{\alpha_0^2}{2} x^2 + \frac{\alpha_0}{2} t x + y.$$

Here x and y denote unknown complex variables and α_0, α_1 are complex parameters satisfying the relation:

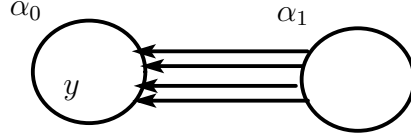
$$(39) \quad 2\alpha_0 + \alpha_1 = 1.$$

By putting $q := x$, we obtain the following equation:

$$\frac{d^2 q}{dt^2} = \left(\frac{2}{q}\right) \left(\frac{dq}{dt}\right)^2 - \alpha_0 q^2 - tq + \frac{q^2}{2} - \frac{2}{q}.$$

Theorem 4.1. *The system (37) admits affine Weyl group symmetry of type $A_2^{(2)}$ as the group of its Bäcklund transformations (cf. [6]), whose generators are explicitly given as follows:*

$$\begin{aligned} s_0 : (x, y, t; \alpha_0, \alpha_1) &\rightarrow (x + \frac{\alpha_0}{y}, y, t; -\alpha_0, \alpha_1 + 4\alpha_0), \\ s_1 : (x, y, t; \alpha_0, \alpha_1) &\rightarrow (-x, -y + \frac{\alpha_1}{x} - \frac{t}{x^2} - \frac{2}{x^4}, t; \alpha_0 + \alpha_1, -\alpha_1). \end{aligned}$$



Dynkin diagram of type $A_2^{(2)}$
 FIGURE 9. Dynkin diagram of type $A_2^{(2)}$

Theorem 4.2. *Let us consider a polynomial Hamiltonian system with Hamiltonian $H \in \mathbb{C}(t)[x, y]$. We assume that*

(A1) $\deg(H) = 6$ with respect to x, y .

(A2) *This system becomes again a polynomial Hamiltonian system in each coordinate r_i ($i = 0, 1$)(cf. [3]):*

$$\begin{aligned} r_0 : x_0 &= \frac{1}{x}, \quad y_0 = -(yx + \alpha_0)x, \\ r_1 : x_1 &= x, \quad y_1 = y - \frac{\alpha_1}{x} + \frac{t}{x^2} + \frac{2}{x^4}. \end{aligned}$$

Then such a system coincides with the system (37).

Theorems 4.1, 4.2 can be checked by a direct calculation, respectively.

Theorem 4.3. *For the system (18), we make the change of parameters and variables*

$$(40) \quad \alpha_0 = \frac{1}{4\varepsilon^6}, \quad \alpha_1 = A_0, \quad \alpha_2 = \frac{4A_1\varepsilon^6 - 1}{12\varepsilon^6}, \quad t = -\frac{1 - \varepsilon^4 T}{\sqrt{2}\varepsilon^3},$$

$$(41) \quad x = \frac{\sqrt{2}\varepsilon^3 X}{2\varepsilon^2 + X}, \quad y = \frac{(X + 2\varepsilon^2)(XY + 2\varepsilon^2 Y + A_0)}{2\sqrt{2}\varepsilon^5}$$

from $\alpha_0, \alpha_1, \alpha_2, t, x, y$ to $A_0, A_1, \varepsilon, T, X, Y$. Then the system (18) can also be written in the new variables T, X, Y and parameters A_0, A_1, ε as a Hamiltonian system. This new system tends to the system (37) as $\varepsilon \rightarrow 0$.

By proving the following theorem, we see how the degeneration process given in Theorem 4.3 works on the Bäcklund transformation group $W(G_2^{(1)})$ (cf. [17]).

Theorem 4.4. *For the degeneration process in Theorem 4.3, we can choose a subgroup $W_{G_2^{(1)} \rightarrow A_2^{(2)}}$ of the Bäcklund transformation group $W(G_2^{(1)})$ so that $W_{G_2^{(1)} \rightarrow A_2^{(2)}}$ converges to the Bäcklund transformation group $W(A_2^{(2)})$ of the system (37) as $\varepsilon \rightarrow 0$.*

Proof of Theorem 4.4. Notice that

$$2A_0 + A_1 = \alpha_0 + 2\alpha_1 + 3\alpha_2 = 1$$

and the change of variables from (x, y) to (X, Y) is symplectic. Since the change of parameters (40) is not one to one, we consider the degeneration process by introducing formal power series of the new parameters A_0, A_1, ε .

We choose S_0, S_1 as

$$S_0 := s_1, \quad S_1 := s_0 s_2$$

and put $W_{G_2^{(1)} \rightarrow A_2^{(2)}} = \langle S_0, S_1 \rangle$. Notice the S_0, S_1 are reflections of the parameters A_0, A_1 , respectively.

Then we can obtain

$$\begin{aligned} S_0(A_0, A_1, \varepsilon) &= (-A_0, A_1 + 4A_0, \varepsilon(1 + 4A_0\varepsilon^6)^{-1/6}), \\ S_1(A_0, A_1, \varepsilon) &= (A_0 + A_1, -A_1, -\sqrt{-1}\varepsilon). \end{aligned}$$

Here, we make the same convention as in Section 2 that $(1 + 4A_0\varepsilon^6)^{-1/6}$ means formal power series of $A_0\varepsilon^6$ with 1 as constant term.

Then we can verify

$$\begin{aligned} S_0(X, Y, T) &= (X + \frac{A_0}{Y}, Y, T), \\ S_1(X, Y, T) &= (-X, -Y + \frac{A_1}{X} - \frac{T}{X^2} - \frac{2}{X^4}, T) \end{aligned}$$

as $\varepsilon \rightarrow 0$.

The proof has thus been completed. \square

By the following theorem, we will show that the system (37) coincides with the system of type $A_1^{(1)}$.

Theorem 4.5. *For the system (37), we make the change of parameters and variables*

$$(42) \quad \beta_0 = \alpha_0 + \alpha_1, \quad \beta_1 = \alpha_0,$$

$$(43) \quad X = \frac{1}{x}, \quad Y = -x(xy + \alpha_0), \quad T = t$$

from $x, y, t, \alpha_0, \alpha_1$ to $X, Y, t, \beta_0, \beta_1$. Then this new system coincides with the system of type $A_1^{(1)}$ (see [6]):

$$(44) \quad \begin{cases} \frac{dX}{dT} = -X^2 + Y - \frac{T}{2}, \\ \frac{dY}{dT} = 2XY + \beta_1 \end{cases}$$

with the Hamiltonian

$$(45) \quad H_{A_1^{(1)}}(X, Y, T; \beta_0, \beta_1) = \frac{1}{2}Y^2 - \left(X^2 + \frac{T}{2}\right)Y - \beta_1X.$$

By putting $q = X$, we have the second Painlevé equation (see [6]):

$$\frac{d^2q}{dT^2} = 2q^3 + Tq + \left(\beta_1 - \frac{1}{2}\right).$$

Proof of Theorem 4.5. Notice that

$$(46) \quad 2\alpha_0 + \alpha_1 = \beta_0 + \beta_1 = 1$$

and the change of variables from (x, y, t) to (X, Y, t) in Theorem 4.5 is symplectic. Choose S_1 and π as

$$S_1 := s_0, \quad \pi := s_1.$$

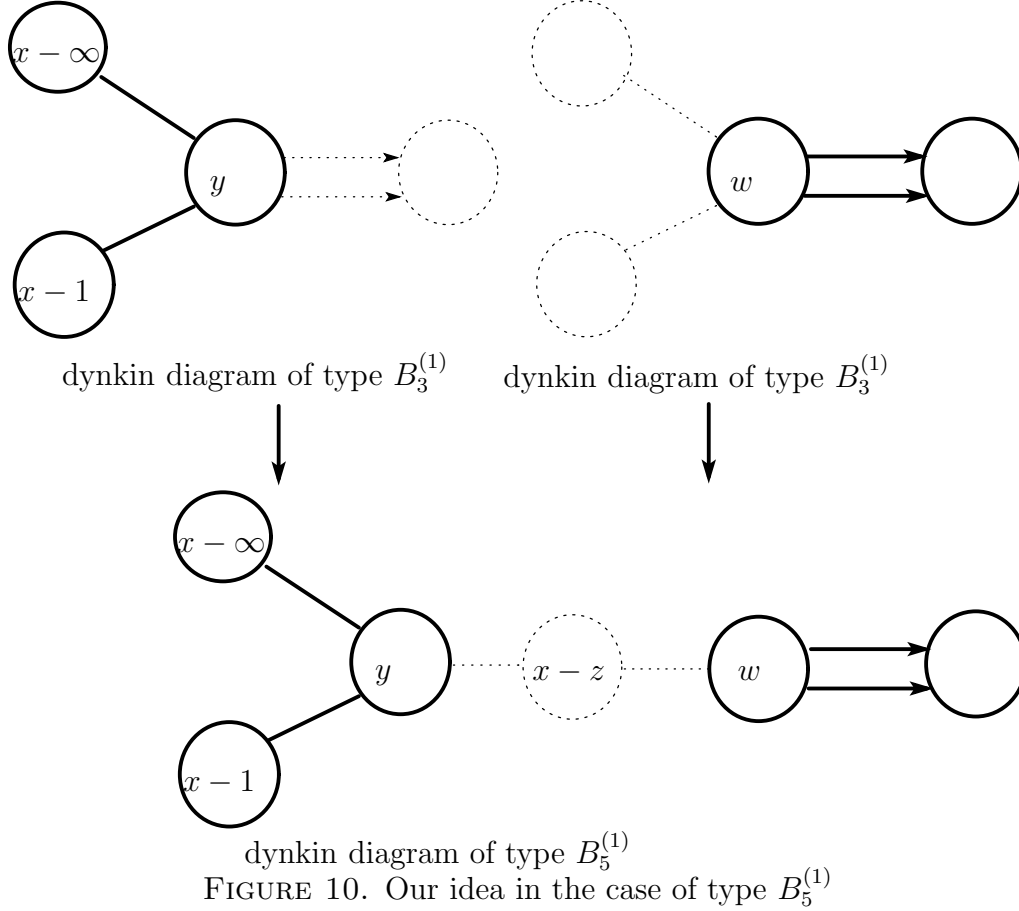
The transformations S_1, π are reflections of

$$\beta_0 = \alpha_0 + \alpha_1, \quad \beta_1 = \alpha_0, \text{ respectively.}$$

We can verify

$$\begin{aligned} S_1 : (X, Y, t; \beta_0, \beta_1) &\rightarrow (X + \frac{\beta_1}{Y}, Y, t; \beta_0 + 2\beta_1, -\beta_1), \\ \pi : (X, Y, t; \beta_0, \beta_1) &\rightarrow (-X, t + 2X^2 - Y, t; \beta_1, \beta_0). \end{aligned}$$

The proof has thus been completed. \square



5. THE CASE OF $B_5^{(1)}$

In this section and next section, we present polynomial Hamiltonian systems in dimension four with affine Weyl group symmetry of types $B_5^{(1)}$ and $D_5^{(2)}$. Our idea is the following way:

- (1) We make a dynkin diagram given by connecting two dynkin diagrams of type $B_3^{(1)}$ (resp. $D_3^{(2)}$) by adding the term with invariant divisor $x - z$.
- (2) We make the symmetry associated with the dynkin diagram given by 1.
- (3) We make the holomorphy conditions r_i associated with the symmetry given by 2.
- (4) We look for a polynomial Hamiltonian system with the holomorphy conditions given by 3, that is,

Problem 5.1. *Let us consider a polynomial Hamiltonian system with Hamiltonian $H \in \mathbb{C}(t)[x, y, z, w]$. We assume that*

(A1) *$\deg(H) = 5$ with respect to x, y, z, w .*

(A2) *This system becomes again a polynomial Hamiltonian system in each coordinate r_i given in 4.*

To solve Problem 5.1, for the Hamiltonian satisfying the assumption (A1) we only have to determine unknown coefficients so that they cancel the poles of Hamiltonian transformed by each r_i .

In this section, we present a 5-parameter family of polynomial Hamiltonian systems that can be considered as four-dimensional coupled Painlevé V systems, which is explicitly given by

$$(47) \quad \left\{ \begin{aligned} \frac{dx}{dt} &= -\frac{2x^3y - 2x^2y + (\alpha_1 + 2\alpha_2)x^2 + (t - 1 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5)x - t}{t} \\ &\quad - \frac{2(x-1)z(zw + \alpha_4)}{t}, \\ \frac{dy}{dt} &= \frac{3x^2y^2 - 2xy^2 + 2(\alpha_1 + 2\alpha_2)xy + (t - 1 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5)y + \alpha_2(\alpha_1 + \alpha_2)}{t} \\ &\quad + \frac{2yz(zw + \alpha_4)}{t}, \\ \frac{dz}{dt} &= -\frac{2z^3w - 2z^2w + (\alpha_1 + 2\alpha_2 + \alpha_3 + 2\alpha_4)z^2 + (t - 1 + 2\alpha_5)z - t}{t} \\ &\quad - \frac{2(x-1)yz^2}{t}, \\ \frac{dw}{dt} &= \frac{3z^2w^2 - 2zw^2 + 2(\alpha_1 + 2\alpha_2 + \alpha_3 + 2\alpha_4)zw + (t - 1 + 2\alpha_5)w}{t} \\ &\quad + \frac{\alpha_4(\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4) + 2(x-1)y(2zw + \alpha_4)}{t} \end{aligned} \right.$$

with the Hamiltonian

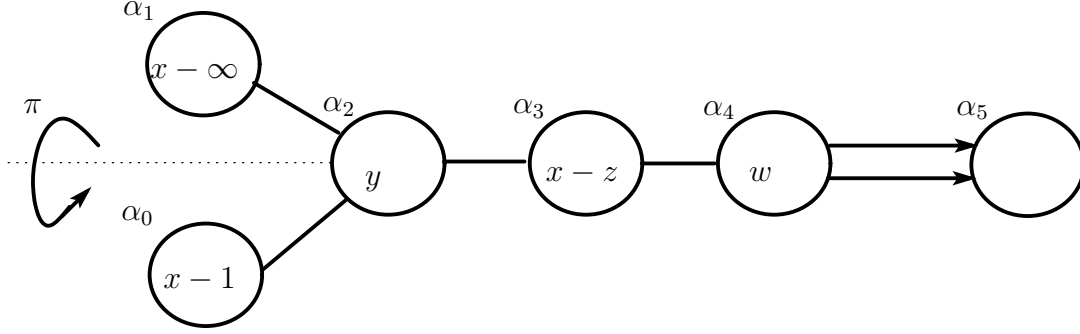
$$(48) \quad \begin{aligned} &H_{B_5^{(1)}}(x, y, z, w, t; \alpha_0, \alpha_1, \dots, \alpha_5) \\ &= H_{B_3^{(1)}}(x, y, t; \alpha_1, \alpha_2, \alpha_3 + \alpha_4 + \alpha_5) + H_{B_3^{(1)}}(z, w, t; \alpha_1 + 2\alpha_2 + \alpha_3, \alpha_4, \alpha_5) \\ &\quad - \frac{2(x-1)yz(zw + \alpha_4)}{t}. \end{aligned}$$

Here x, y, z and w denote unknown complex variables, and $\alpha_0, \alpha_1, \dots, \alpha_5$ are complex parameters satisfying the relation:

$$(49) \quad \alpha_0 + \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 = 1.$$

Theorem 5.1. *The system (47) admits extended affine Weyl group symmetry of type $B_5^{(1)}$ as the group of its Bäcklund transformations (cf. [6]), whose generators are explicitly given as follows: with the notation*

$$(*) := (x, y, z, w, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5),$$

FIGURE 11. Dynkin diagram of type $B_5^{(1)}$

$$\begin{aligned}
s_0 : (*) &\rightarrow (x, y - \frac{\alpha_0}{x-1}, z, w, t; -\alpha_0, \alpha_1, \alpha_2 + \alpha_0, \alpha_3, \alpha_4, \alpha_5), \\
s_1 : (*) &\rightarrow (x, y, z, w, t; \alpha_0, -\alpha_1, \alpha_2 + \alpha_1, \alpha_3, \alpha_4, \alpha_5), \\
s_2 : (*) &\rightarrow (x + \frac{\alpha_2}{y}, y, z, w, t; \alpha_0 + \alpha_2, \alpha_1 + \alpha_2, -\alpha_2, \alpha_3 + \alpha_2, \alpha_4, \alpha_5), \\
s_3 : (*) &\rightarrow (x, y - \frac{\alpha_3}{x-z}, z, w + \frac{\alpha_3}{x-z}, t; \alpha_0, \alpha_1, \alpha_2 + \alpha_3, -\alpha_3, \alpha_4 + \alpha_3, \alpha_5), \\
s_4 : (*) &\rightarrow (x, y, z + \frac{\alpha_4}{w}, w, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3 + \alpha_4, -\alpha_4, \alpha_5 + \alpha_4), \\
s_5 : (*) &\rightarrow (x, y, z, w - \frac{2\alpha_5}{z} + \frac{t}{z^2}, -t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4 + 2\alpha_5, -\alpha_5), \\
\pi : (*) &\rightarrow (\frac{x}{x-1}, -(x-1)((x-1)y + \alpha_2), \frac{z}{z-1}, -(z-1)((z-1)w + \alpha_4), -t; \\
&\quad \alpha_1, \alpha_0, \alpha_2, \alpha_3, \alpha_4, \alpha_5).
\end{aligned}$$

Theorem 5.2. *Let us consider a polynomial Hamiltonian system with Hamiltonian $H \in \mathbb{C}(t)[x, y, z, w]$. We assume that*

(A1) *$\deg(H) = 5$ with respect to x, y, z, w .*

(A2) *This system becomes again a polynomial Hamiltonian system in each coordinate r_i ($i = 0, 1, \dots, 5$):*

$$\begin{aligned}
r_0 : x_0 &= -((x-1)y - \alpha_0)y, \quad y_0 = \frac{1}{y}, \quad z_0 = z, \quad w_0 = w, \\
r_1 : x_1 &= \frac{1}{x}, \quad y_1 = -(yx + \alpha_1 + \alpha_2)x, \quad z_1 = z, \quad w_1 = w, \\
r_2 : x_2 &= \frac{1}{x}, \quad y_2 = -(yx + \alpha_2)x, \quad z_2 = z, \quad w_2 = w, \\
r_3 : x_3 &= -((x-z)y - \alpha_3)y, \quad y_3 = \frac{1}{y}, \quad z_3 = z, \quad w_3 = w + y, \\
r_4 : x_4 &= x, \quad y_4 = y, \quad z_4 = \frac{1}{z}, \quad w_4 = -(wz + \alpha_4)z, \\
r_5 : x_5 &= x, \quad y_5 = y, \quad z_5 = z, \quad w_5 = w - \frac{2\alpha_5}{z} + \frac{t}{z^2}.
\end{aligned}$$

Then such a system coincides with the system (47).

By the following theorem, we will show that the system (47) coincides with a 5-parameter family of four-dimensional coupled Painlevé V systems with the affine Weyl group symmetry of type $D_5^{(1)}$ (see [14]).

Theorem 5.3. *For the system (47), we make the change of parameters and variables*

(50)

$$\beta_0 = \alpha_4 + 2\alpha_5, \quad \beta_1 = \alpha_4, \quad \beta_2 = \alpha_3, \quad \beta_3 = \alpha_2, \quad \beta_4 = \alpha_0, \quad \beta_5 = \alpha_1,$$

(51)

$$X = \frac{1}{z}, \quad Y = -z(zw + \alpha_4), \quad Z = \frac{1}{x}, \quad W = -x(xy + \alpha_2), \quad T = -t$$

from $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, t, x, y, z, w$ to $\beta_0, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5, T, X, Y, Z, W$. Then the system (47) can also be written in the new variables T, X, Y, Z, W and parameters $\beta_0, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5$ as a Hamiltonian system. This new system coincides with the system of type $D_5^{(1)}$:

(52)

$$\left\{ \begin{array}{l} \frac{dX}{dT} = \frac{2X^2Y}{T} + X^2 - \frac{2XY}{T} - (1 + \frac{2\beta_2 + 2\beta_3 + \beta_5 + \beta_4}{T})X + \frac{\beta_2 + \beta_5}{T} \\ \quad + \frac{2Z((Z-1)W + \beta_3)}{T}, \\ \frac{dY}{dT} = -\frac{2XY^2}{T} + \frac{Y^2}{T} - 2XY + (1 + \frac{2\beta_2 + 2\beta_3 + \beta_5 + \beta_4}{T})Y - \beta_1, \\ \frac{dZ}{dT} = \frac{2Z^2W}{T} + Z^2 - \frac{2ZW}{T} - (1 + \frac{\beta_5 + \beta_4}{T})Z + \frac{\beta_5}{T} + \frac{2YZ(Z-1)}{T}, \\ \frac{dW}{dT} = -\frac{2ZW^2}{T} + \frac{W^2}{T} - 2ZW + (1 + \frac{\beta_5 + \beta_4}{T})W - \beta_3 - \frac{2Y(-W + 2ZW + \beta_3)}{T} \end{array} \right.$$

with the Hamiltonian

(53)

$$H = H_{A_3^{(1)}}(X, Y, T; \beta_2 + \beta_5, \beta_1, \beta_2 + 2\beta_3 + \beta_4) + H_{A_3^{(1)}}(Z, W, T; \beta_5, \beta_3, \beta_4) \\ + \frac{2YZ\{(Z-1)W + \beta_3\}}{T}.$$

Proof of Theorem 5.3. Notice that

$$\begin{aligned} & \alpha_0 + \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 \\ &= \beta_0 + \beta_1 + 2\beta_2 + 2\beta_3 + \beta_4 + \beta_5 = 1 \end{aligned}$$

and the change of variables from (x, y, z, w, t) to (X, Y, Z, W, T) in Theorem 5.3 is symplectic. Choose S_i , $i = 0, 1, \dots, 8$ as

$$\begin{aligned} S_0 &:= s_5 s_4 s_5, & S_1 &:= s_4, & S_2 &:= s_3, & S_3 &:= s_2, & S_4 &:= s_0, \\ S_5 &:= s_1, & S_6 &:= s_5, & S_7 &:= \pi, & S_8 &:= \pi s_5. \end{aligned}$$

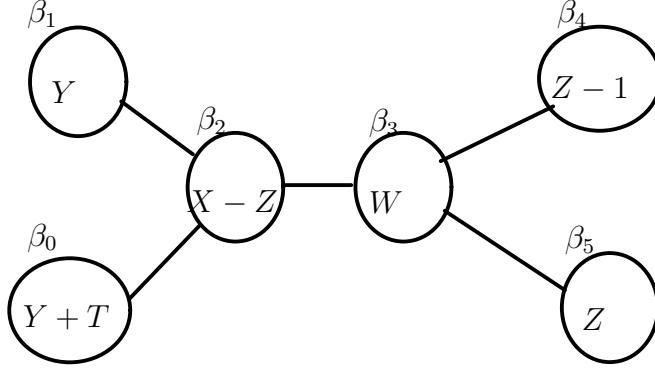
The transformations S_0, S_1, \dots, S_5 are reflections of

$$\beta_0 = \alpha_4 + 2\alpha_5, \quad \beta_1 = \alpha_4, \quad \beta_2 = \alpha_3, \quad \beta_3 = \alpha_2, \quad \beta_4 = \alpha_0, \quad \beta_5 = \alpha_1 \text{ respectively.}$$

By using the notation

$$(*) := (X, Y, Z, W, T; \beta_0, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5),$$

we can verify

FIGURE 12. Dynkin diagram of type $B_5^{(1)}$

$$\begin{aligned}
S_0 : (*) &\rightarrow (X + \frac{\beta_0}{Y+T}, Y, Z, W, T; -\beta_0, \beta_1, \beta_2 + \beta_0, \beta_3, \beta_4, \beta_5), \\
S_1 : (*) &\rightarrow (X + \frac{\beta_1}{Y}, Y, Z, W, T; \beta_0, -\beta_1, \beta_2 + \beta_1, \beta_3, \beta_4, \beta_5), \\
S_2 : (*) &\rightarrow (X, Y - \frac{\beta_2}{X-Z}, Z, W + \frac{\beta_2}{X-Z}, T; \beta_0 + \beta_2, \beta_1 + \beta_2, -\beta_2, \beta_3 + \beta_2, \beta_4, \beta_5), \\
S_3 : (*) &\rightarrow (X, Y, Z + \frac{\beta_3}{W}, W, T; \beta_0, \beta_1, \beta_2 + \beta_3, -\beta_3, \beta_4 + \beta_3, \beta_5 + \beta_3), \\
S_4 : (*) &\rightarrow (X, Y, Z, W - \frac{\beta_4}{Z-1}, T; \beta_0, \beta_1, \beta_2, \beta_3 + \beta_4, -\beta_4, \beta_5), \\
S_5 : (*) &\rightarrow (X, Y, Z, W - \frac{\beta_5}{Z}, T; \beta_0, \beta_1, \beta_2, \beta_3 + \beta_5, \beta_4, -\beta_5), \\
S_6 : (*) &\rightarrow (X, Y + T, Z, W, -T; \beta_1, \beta_0, \beta_2, \beta_3, \beta_4, \beta_5), \\
S_7 : (*) &\rightarrow (1 - X, -Y, 1 - Z, -W, -T; \beta_0, \beta_1, \beta_2, \beta_3, \beta_5, \beta_4), \\
S_8 : (*) &\rightarrow (1 - X, -Y - T, 1 - Z, -W, T; \beta_1, \beta_0, \beta_2, \beta_3, \beta_5, \beta_4).
\end{aligned}$$

The transformations S_i , $i = 0, 1, \dots, 8$, define a representation of the affine Weyl group of type $D_5^{(1)}$, that is, they satisfy the following relations:

$$\begin{aligned}
S_0^2 &= S_1^2 = \dots = S_5^2 = S_6^2 = S_7^2 = S_8^2 = 1, \\
(S_0 S_1)^2 &= (S_0 S_3)^2 = (S_0 S_4)^2 = (S_0 S_5)^2 = (S_1 S_3)^2 \\
&= (S_1 S_4)^2 = (S_1 S_5)^2 = (S_2 S_4)^2 = (S_2 S_5)^2 = (S_4 S_5)^2 = 1, \\
(S_0 S_2)^3 &= (S_1 S_2)^3 = (S_2 S_3)^3 = (S_3 S_4)^3 = (S_3 S_5)^3 = 1, \\
S_6(S_0, S_1, S_2, S_3, S_4, S_5) &= (S_1, S_0, S_2, S_3, S_4, S_5) S_6, \\
S_7(S_0, S_1, S_2, S_3, S_4, S_5) &= (S_0, S_1, S_2, S_3, S_5, S_4) S_7, \\
S_8(S_0, S_1, S_2, S_3, S_4, S_5) &= (S_1, S_0, S_2, S_3, S_5, S_4) S_8.
\end{aligned}$$

The proof has thus been completed. \square

Proposition 5.1. *The system (47) admits the following transformation as its Bäcklund transformation:*

$$\begin{aligned} \varphi : (x, y, z, w, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \rightarrow \\ \left(\frac{t}{(t + z^2w + \alpha_4z)}, \frac{(t + z^2w + \alpha_4z)(-tx + tz - xz^2w + z^3w - (\alpha_3 + \alpha_4)xz + \alpha_4z^2)}{txz}, \right. \\ \left. \frac{t}{(t + x^2y + z^2w + \alpha_2x + \alpha_4z)}, \right. \\ \left. \frac{(t + x^2y + z^2w + \alpha_2x + \alpha_4z)\{(x - t)(t + x^2y + z^2w + \alpha_4z) - (\alpha_0 + \alpha_2)x + \alpha_2x^2\}}{tx}, \right. \\ \left. -t; \alpha_4, \alpha_4 + 2\alpha_5, \alpha_3, \alpha_2, \alpha_0, \frac{\alpha_1 - \alpha_0}{2} \right). \end{aligned}$$

We note that this transformation φ is pulled back the following diagram automorphism π of the system (52) by the symplectic transformation defined in Theorem 5.3

$$\begin{aligned} \pi : (X, Y, Z, W, T; \beta_0, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5) \rightarrow \\ ((Y + W + T)/T, -T(Z - 1), (Y + T)/T, -T(X - Z), -T; \beta_5, \beta_4, \beta_3, \beta_2, \beta_1, \beta_0). \end{aligned}$$

6. THE SYSTEM OF $D_5^{(2)}$

In this section, we present a 4-parameter family of polynomial Hamiltonian systems that can be considered as four-dimensional coupled Painlevé III systems, which is given as follows

$$(54) \quad \begin{cases} \frac{dx}{dt} = \frac{2x^2y}{t} - x^2 - \frac{2(\alpha_2 + \alpha_3 + \alpha_4)x}{t} + \frac{1}{t} + \frac{2z(zw + \alpha_3)}{t}, \\ \frac{dy}{dt} = -\frac{2xy^2}{t} + 2xy + \frac{2(\alpha_2 + \alpha_3 + \alpha_4)y}{t} + \alpha_1, \\ \frac{dz}{dt} = \frac{2z^2w}{t} - z^2 - \frac{2\alpha_4z}{t} + \frac{1}{t} + \frac{2yz^2}{t}, \\ \frac{dw}{dt} = -\frac{2zw^2}{t} + 2zw + \frac{2\alpha_4w}{t} + \alpha_3 - \frac{2y(2zw + \alpha_3)}{t} \end{cases}$$

with the Hamiltonian

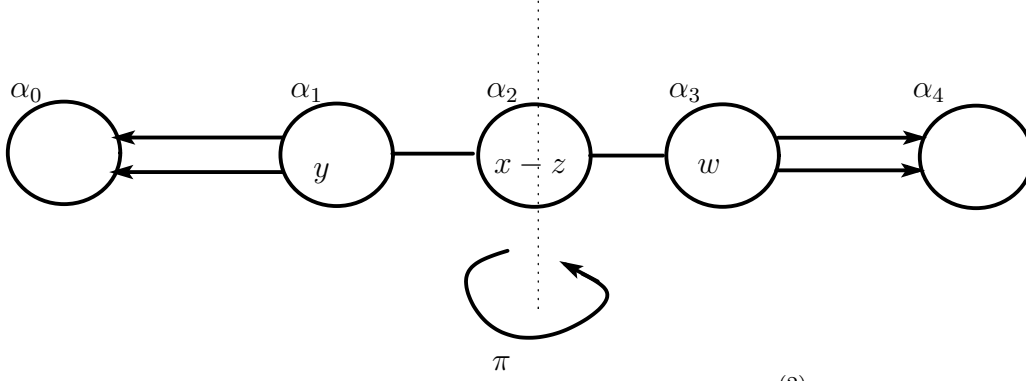
$$\begin{aligned} (55) \quad & H_{D_5^{(2)}}(x, y, z, w, t; \alpha_0, \alpha_1, \dots, \alpha_4) \\ &= H_{D_3^{(2)}}(x, y, t; \alpha_2 + \alpha_3 + \alpha_4, \alpha_1, -\frac{1}{2} + \alpha_0) + H_{D_3^{(2)}}(z, w, t; \alpha_4, \alpha_3, \frac{1}{2} - \alpha_3 - \alpha_4) \\ &+ \frac{2yz(zw + \alpha_3)}{t}. \end{aligned}$$

Here x, y, z and w denote unknown complex variables, and $\alpha_0, \alpha_1, \dots, \alpha_4$ are complex parameters satisfying the relation:

$$(56) \quad \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = \frac{1}{2}.$$

Theorem 6.1. *The system (54) admits extended affine Weyl group symmetry of type $D_5^{(2)}$ as the group of its Bäcklund transformations (cf. [6]), whose generators are explicitly given as follows: with the notation*

$$(*) := (x, y, z, w, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4),$$

FIGURE 13. Dynkin diagram of type $D_5^{(2)}$

$$\begin{aligned}
s_0 : (*) &\rightarrow (x, y - t, z, w, -t; -\alpha_0, \alpha_1 + 2\alpha_0, \alpha_2, \alpha_3, \alpha_4), \\
s_1 : (*) &\rightarrow (x + \frac{\alpha_1}{y}, y, z, w, t; \alpha_0 + \alpha_1, -\alpha_1, \alpha_2 + \alpha_1, \alpha_3, \alpha_4), \\
s_2 : (*) &\rightarrow (x, y - \frac{\alpha_2}{x - z}, z, w + \frac{\alpha_2}{x - z}, t; \alpha_0, \alpha_1 + \alpha_2, -\alpha_2, \alpha_3 + \alpha_2, \alpha_4), \\
s_3 : (*) &\rightarrow (x, y, z + \frac{\alpha_3}{w}, w, t; \alpha_0, \alpha_1, \alpha_2 + \alpha_3, -\alpha_3, \alpha_4 + \alpha_3), \\
s_4 : (*) &\rightarrow (-x, -y, -z, -w + \frac{2\alpha_4}{z} - \frac{1}{z^2}, -t; \alpha_0, \alpha_1, \alpha_2, \alpha_3 + 2\alpha_4, -\alpha_4), \\
\pi : (*) &\rightarrow (\frac{1}{tz}, -t(zw + \alpha_3)z, \frac{1}{tx}, -t(xy + \alpha_1)x, t; \alpha_4, \alpha_3, \alpha_2, \alpha_1, \alpha_0).
\end{aligned}$$

Theorem 6.2. *Let us consider a polynomial Hamiltonian system with Hamiltonian $H \in \mathbb{C}(t)[x, y, z, w]$. We assume that*

(A1) *$\deg(H) = 5$ with respect to x, y, z, w .*

(A2) *This system becomes again a polynomial Hamiltonian system in each coordinate r_i ($i = 0, 1, \dots, 4$):*

$$\begin{aligned}
r_0 : x_0 &= \frac{1}{x}, \quad y_0 = -((y - t)x + 2\alpha_0 + \alpha_1)x, \quad z_0 = z, \quad w_0 = w, \\
r_1 : x_1 &= \frac{1}{x}, \quad y_1 = -(yx + \alpha_1)x, \quad z_1 = z, \quad w_1 = w, \\
r_2 : x_2 &= -((x - z)y - \alpha_2)y, \quad y_2 = \frac{1}{y}, \quad z_2 = z, \quad w_2 = w + y, \\
r_3 : x_3 &= x, \quad y_3 = y, \quad z_3 = \frac{1}{z}, \quad w_3 = -(wz + \alpha_3)z, \\
r_4 : x_4 &= x, \quad y_4 = y, \quad z_4 = z, \quad w_4 = w - \frac{2\alpha_4}{z} + \frac{1}{z^2}.
\end{aligned}$$

Then such a system coincides with the system (54).

Theorem 6.3. *For the system (47), we make the change of parameters and variables:*

(57)

$$\begin{aligned}
x &= \frac{\varepsilon TX}{1 + \varepsilon TX}, \quad y = \frac{(1 + \varepsilon TX)(\varepsilon TXY + Y + A_1 \varepsilon T)}{\varepsilon T}, \quad z = \frac{\varepsilon TZ}{1 + \varepsilon TZ}, \\
w &= \frac{(1 + \varepsilon TZ)(\varepsilon TZW + W + A_3 \varepsilon T)}{\varepsilon T}, \quad t = \varepsilon T,
\end{aligned}$$

(58)

$$\alpha_0 = -\frac{1}{\varepsilon} + 2A_0, \quad \alpha_1 = \frac{1}{\varepsilon}, \quad \alpha_2 = A_1, \quad \alpha_3 = A_2, \quad \alpha_4 = A_3, \quad \alpha_5 = A_4$$

from $x, y, z, w, t, \alpha_0, \alpha_1, \dots, \alpha_5$ to $X, Y, Z, W, T, A_0, A_1, \dots, A_4, \varepsilon$. Then the system (47) can also be written in the new variables T, X, Y, Z, W and parameters $A_0, A_1, \dots, A_4, \varepsilon$ as a Hamiltonian system. This new system tends to the system (54) as $\varepsilon \rightarrow 0$.

By proving the following theorem, we see how the degeneration process given in Theorem 6.3 works on the Bäcklund transformation group $W(B_5^{(1)})$ (cf. [17]).

Theorem 6.4. *For the degeneration process in Theorem 6.3, we can choose a subgroup $W_{B_5^{(1)} \rightarrow D_5^{(2)}}$ of the Bäcklund transformation group $W(B_5^{(1)})$ so that $W_{B_5^{(1)} \rightarrow D_5^{(2)}}$ converges to the Bäcklund transformation group $W(D_5^{(2)})$ of the system (54) as $\varepsilon \rightarrow 0$.*

Proof of Theorem 6.4. Note that

$$2(A_0 + A_1 + A_2 + A_3 + A_4) = \alpha_0 + \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 = 1$$

and the change of variables from (x, y, z, w) to (X, Y, Z, W) is symplectic. Choose S_i ($i = 0, 1, \dots, 4$) as

$$S_0 := s_0 s_1, \quad S_1 := s_2, \quad S_2 := s_3, \quad S_3 = s_4, \quad S_4 = s_5.$$

Then the transformations S_i are reflections of the parameters A_0, A_1, \dots, A_4 . The transformation group $\langle S_0, S_1, \dots, S_4 \rangle$ coincides with the transformations given in Theorem 6.1. The proof has thus been completed. \square

By the following theorem, we will show that the system (54) coincides with a 4-parameter family of four-dimensional coupled Painlevé III systems with the affine Weyl group symmetry of type $B_4^{(1)}$ (see [14]).

Theorem 6.5. *For the system (54), we first make the change of parameters and variables*

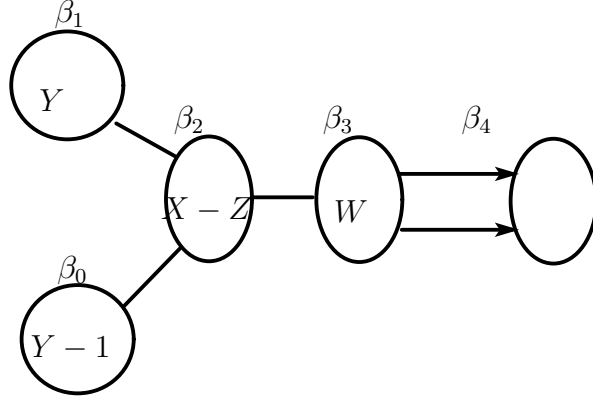
$$(59) \quad \beta_0 = \alpha_3 + 2\alpha_4, \quad \beta_1 = \alpha_3, \quad \beta_2 = \alpha_2, \quad \beta_3 = \alpha_1, \quad \beta_4 = \alpha_0,$$

(60)

$$X = \frac{1}{z}, \quad Y = -z(zw + \alpha_3), \quad Z = \frac{1}{x}, \quad W = -x(xy + \alpha_1), \quad T = t$$

from $\alpha_0, \alpha_1, \dots, \alpha_4, t, x, y, z, w$ to $\beta_0, \beta_1, \dots, \beta_4, T, X, Y, Z, W$. Then the system (54) can also be written in the new variables T, X, Y, Z, W and parameters $\beta_0, \beta_1, \dots, \beta_4$ as a Hamiltonian system. This new system coincides with the system of type $B_4^{(1)}$:

$$(61) \quad \begin{cases} \frac{dX}{dT} = \frac{2X^2Y - X^2 + (\beta_0 + \beta_1)X + 2\beta_3Z + 2Z^2W}{T} + 1, \\ \frac{dY}{dT} = \frac{-2XY^2 + 2XY - (\beta_0 + \beta_1)Y + \beta_1}{T}, \\ \frac{dZ}{dT} = \frac{2Z^2W - Z^2 + (1 - 2\beta_4)Z + 2YZ^2}{T} + 1, \\ \frac{dW}{dT} = \frac{-2ZW^2 + 2ZW - (1 - 2\beta_4)W - 2\beta_3Y - 4YZW + \beta_3}{T} \end{cases}$$

FIGURE 14. Dynkin diagram of type $B_4^{(1)}$

with the Hamiltonian

(62)

$$\begin{aligned}
 H &= H_{C_2^{(1)}}(X, Y, T; \beta_0, \frac{1 - \beta_0 - \beta_1}{2}, \beta_1) + H_{C_2^{(1)}}(Z, W, T; \beta_0 + \beta_1 + 2\beta_2 + \beta_3, \beta_4, \beta_3) \\
 &\quad + \frac{2YZ(ZW + \beta_3)}{T} \\
 &= \frac{X^2Y(Y - 1) + X\{(\beta_0 + \beta_1)Y - \beta_1\} + TY}{T} \\
 &\quad + \frac{Z^2W(W - 1) + Z\{(1 - 2\beta_4)W - \beta_3\} + TW}{T} + \frac{2YZ(ZW + \beta_3)}{T}.
 \end{aligned}$$

Proof of Theorem 6.5. Notice that

$$2(\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) = \beta_0 + \beta_1 + 2\beta_2 + 2\beta_3 + 2\beta_4 = 1$$

and the change of variables from (x, y, z, w, t) to (X, Y, Z, W, T) in Theorem 6.5 is symplectic. Choose S_i , $i = 0, 1, \dots, 6$ as

$$\begin{aligned}
 S_0 &:= s_4 s_3 s_4, & S_1 &:= s_3, & S_2 &:= s_2, \\
 S_3 &:= s_1, & S_4 &:= s_0, & S_5 &:= s_4, & S_6 &:= \pi.
 \end{aligned}$$

The transformations S_0, S_1, \dots, S_4 are reflections of

$$\beta_0 = \alpha_3 + 2\alpha_4, \quad \beta_1 = \alpha_3, \quad \beta_2 = \alpha_2, \quad \beta_3 = \alpha_1, \quad \beta_4 = \alpha_0 \text{ respectively.}$$

By using the notation

$$(*) := (X, Y, Z, W, T; \beta_0, \beta_1, \beta_2, \beta_3, \beta_4),$$

we can verify

$$\begin{aligned}
S_0 : (*) &\rightarrow (X + \frac{\beta_0}{Y-1}, Y, Z, W, T; -\beta_0, \beta_1, \beta_2 + \beta_0, \beta_3, \beta_4), \\
S_1 : (*) &\rightarrow (X + \frac{\beta_1}{Y}, Y, Z, W, T; \beta_0, -\beta_1, \beta_2 + \beta_1, \beta_3, \beta_4), \\
S_2 : (*) &\rightarrow (X, Y - \frac{\beta_2}{X-Z}, Z, W + \frac{\beta_2}{X-Z}, T; \beta_0 + \beta_2, \beta_1 + \beta_2, -\beta_2, \beta_3 + \beta_2, \beta_4), \\
S_3 : (*) &\rightarrow (X, Y, Z + \frac{\beta_3}{W}, W, T; \beta_0, \beta_1, \beta_2 + \beta_3, -\beta_3, \beta_4 + \beta_3), \\
S_4 : (*) &\rightarrow (X, Y, Z, W - \frac{2\beta_4}{Z} + \frac{T}{Z^2}, -T; \beta_0, \beta_1, \beta_2, \beta_3 + 2\beta_4, -\beta_4), \\
S_5 : (*) &\rightarrow (-X, 1 - Y, -Z, -W, -T; \beta_1, \beta_0, \beta_2, \beta_3, \beta_4), \\
S_6 : (*) &\rightarrow (\frac{T}{Z}, -\frac{(ZW + \beta_3)Z}{T}, \frac{T}{X}, -\frac{(XY + \beta_1)X}{T}, T; \beta_3 + 2\beta_4, \beta_3, \beta_2, \beta_1, \frac{\beta_0 - \beta_1}{2}).
\end{aligned}$$

The transformations S_i , $i = 0, 1, \dots, 6$, define a representation of the extended affine Weyl group of type $B_4^{(1)}$, that is, they satisfy the following relations:

$$\begin{aligned}
S_0^2 &= S_1^2 = \dots = S_4^2 = S_5^2 = S_6^2 = 1, \\
(S_0 S_1)^2 &= (S_0 S_3)^2 = (S_0 S_4)^2 = (S_1 S_3)^2 = (S_1 S_4)^2 = (S_2 S_4)^2 = 1, \\
(S_0 S_2)^3 &= (S_1 S_2)^3 = (S_2 S_3)^3 = 1, \quad (S_3 S_4)^4 = 1, \\
S_5(S_0, S_1, S_2, S_3, S_4) &= (S_1, S_0, S_2, S_3, S_4) S_5.
\end{aligned}$$

The proof has thus been completed. \square

7. FURTHER PROBLEMS

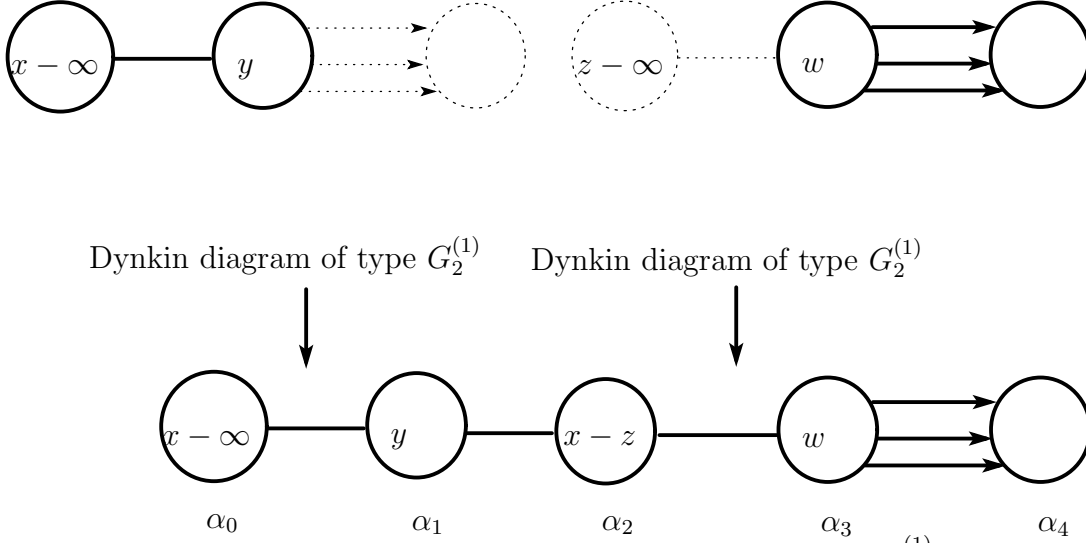
For the system of type $G_2^{(1)}$, let us consider a generalization of this system with the same way in Sections 5 and 6.

By using the coupling term $x - z$, we make the following representation:

$$\begin{aligned}
(63) \quad s_0(*) &\rightarrow (x, y, z, w, t; -\alpha_0, \alpha_1 + \alpha_0, \alpha_2, \alpha_3, \alpha_4), \\
s_1(*) &\rightarrow (x + \frac{\alpha_1}{y}, y, z, w, t; \alpha_0 + \alpha_1, -\alpha_1, \alpha_2 + \alpha_1, \alpha_3, \alpha_4), \\
s_2(*) &\rightarrow (x, y - \frac{\alpha_2}{x-z}, z, w + \frac{\alpha_2}{x-z}, t; \alpha_0, \alpha_1 + \alpha_2, -\alpha_2, \alpha_3 + \alpha_2, \alpha_4), \\
s_3(*) &\rightarrow (x, y, z + \frac{\alpha_3}{w}, w, t; \alpha_0, \alpha_1, \alpha_2 + \alpha_3, -\alpha_3, \alpha_4 + \alpha_3), \\
s_4(*) &\rightarrow (\sqrt{-1}x, -\sqrt{-1}y, \sqrt{-1}z, -\sqrt{-1}(w - \frac{3\alpha_4}{z} + \frac{t}{z^2} + \frac{1}{2z^3}), -\sqrt{-1}t; \\
&\quad \alpha_0, \alpha_1, \alpha_2, \alpha_3 + 3\alpha_4, -\alpha_4).
\end{aligned}$$

Lemma 7.1. *These transformations s_i ($i = 0, 1, \dots, 4$) satisfy the following relations:*

$$\begin{aligned}
(64) \quad s_i^2 &= 1, \quad (s_0 s_2)^2 = (s_0 s_3)^2 = (s_0 s_4)^2 = (s_1 s_3)^2 = (s_1 s_4)^2 = (s_2 s_4)^2 = 1, \\
(s_0 s_1)^3 &= (s_1 s_2)^3 = (s_2 s_3)^3 = 1, \quad (s_3 s_4)^6 = 1.
\end{aligned}$$

FIGURE 15. A generalization of the system of type $G_2^{(1)}$

Let us make the holomorphy conditions associated with (63).

$$\begin{aligned}
 r_0 : x_0 &= \frac{1}{x}, & y_0 &= -(yx + \alpha_0 + \alpha_1)x, & z_0 &= z, & w_0 &= w, \\
 r_1 : x_1 &= \frac{1}{x}, & y_1 &= -(yx + \alpha_1)x, & z_1 &= z, & w_1 &= w, \\
 (65) \quad r_2 : x_2 &= -((x - z)y - \alpha_2)y, & y_2 &= \frac{1}{y}, & z_2 &= z, & w_2 &= w + y, \\
 r_3 : x_3 &= x, & y_3 &= y, & z_3 &= \frac{1}{z}, & w_3 &= -(wz + \alpha_3)z, \\
 r_4 : x_4 &= x, & y_4 &= y, & z_4 &= z, & w_4 &= w - \frac{3\alpha_4}{z} + \frac{t}{z^2} + \frac{1}{2z^3}.
 \end{aligned}$$

Problem 7.1. *Let us consider a polynomial Hamiltonian system with Hamiltonian $H \in \mathbb{C}(t)[x, y, z, w]$. We assume that*

- (A1) $\deg(H) = 5$ with respect to x, y, z, w .
- (A2) *This system becomes again a polynomial Hamiltonian system in each coordinate r_i ($i = 0, 1, \dots, 4$).*

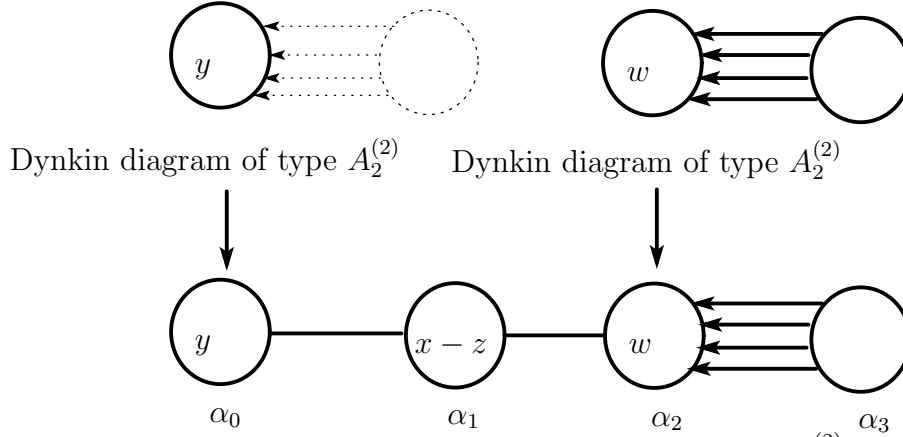
It is a pity that we can not find a polynomial Hamiltonian system satisfying the assumptions (A1) and (A2).

It is still an open question whether we can find a generalization of the system of type $G_2^{(1)}$.

For the system of type $A_2^{(2)}$, let us consider a generalization of this system with the same way in Sections 5 and 6.

By using the coupling term $x - z$, we make the following representation:

$$\begin{aligned}
 s_0(*) &\rightarrow (x + \frac{\alpha_0}{y}, y, z, w, t; -\alpha_0, \alpha_1 + \alpha_0, \alpha_2, \alpha_3), \\
 s_1(*) &\rightarrow (x, y - \frac{\alpha_1}{x - z}, z, w + \frac{\alpha_1}{x - z}, t; \alpha_0 + \alpha_1, -\alpha_1, \alpha_2 + \alpha_1, \alpha_3), \\
 (66) \quad s_2(*) &\rightarrow (x, y, z + \frac{\alpha_2}{w}, w, t; \alpha_0, \alpha_1 + \alpha_2, -\alpha_2, \alpha_3 + 4\alpha_2), \\
 s_3(*) &\rightarrow (-x, -y, -z, -w + \frac{\alpha_3}{z} - \frac{t}{z^2} - \frac{2}{z^4}, t; \alpha_0, \alpha_1, \alpha_2 + \alpha_3, -\alpha_3).
 \end{aligned}$$


 FIGURE 16. A generalization of the system of type $A_2^{(2)}$

Lemma 7.2. *These transformations s_i ($i = 0, 1, 2, 3$) satisfy the following relations:*

$$(67) \quad s_i^2 = 1, \quad (s_0 s_2)^2 = (s_0 s_3)^2 = (s_1 s_3)^2 = 1, \quad (s_0 s_1)^3 = (s_1 s_2)^3 = 1.$$

Let us make the holomorphy conditions associated with (66).

$$(68) \quad \begin{aligned} r_0 : x_0 &= \frac{1}{x}, & y_0 &= -(yx + \alpha_0)x, & z_0 &= z, & w_0 &= w, \\ r_1 : x_1 &= -((x - z)y - \alpha_1)y, & y_1 &= \frac{1}{y}, & z_1 &= z, & w_1 &= w + y, \\ r_2 : x_2 &= x, & y_2 &= y, & z_2 &= \frac{1}{z}, & w_2 &= -(wz + \alpha_2)z, \\ r_3 : x_3 &= x, & y_3 &= y, & z_3 &= z, & w_3 &= w - \frac{\alpha_3}{z} + \frac{t}{z^2} + \frac{2}{z^4}. \end{aligned}$$

Problem 7.2. *Let us consider a polynomial Hamiltonian system with Hamiltonian $H \in \mathbb{C}(t)[x, y, z, w]$. We assume that*

(A1) *$\deg(H) = 6$ with respect to x, y, z, w .*

(A2) *This system becomes again a polynomial Hamiltonian system in each coordinate r_i ($i = 0, 1, 2, 3$).*

It is a pity that we can not find a polynomial Hamiltonian system satisfying the assumptions (A1) and (A2).

It is still an open question whether we can find a generalization of the system of type $A_2^{(2)}$.

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